

1.3 Talk - Tangent Spaces

1.3.1 Intuition

In the Euclidean case (flat spaces or manifolds embedded in the Euclidean space - not the projective space), we can get the tangent vector at a point and the plane associated with it.

Another interpretation, is that if we have a curve $\gamma : (-\epsilon, \epsilon) \rightarrow \mathbb{R}^n$, where $\gamma(0) = p$, the vector v is the velocity of the curve $\gamma'(0) = v$.

1.3.2 Tangent Spaces

Definition 1.1 (Germ). In $\mathbb{R}^n$, a *germ* is an element of $C_p^\infty(M)$, where C^∞ is function at p from a neighborhood U in M to $\mathbb{R}$, which is the equivalence class, where two such functions being equivalent if they agree on some, possibly smaller, neighborhood of p .

Note: The importance of germs is to inspect what happens near p , so that the tangent vector only depends on what happens at p and not the global function.

A derivation at a point is the same as saying "tangent vector", in a sense. The tangent space is the the space of derivations at p , or alternatively the velocities of curves.

Definition 1.2 (Derivation at a point). We define a derivation at a point in a manifold M , or a point-derivation of $C_p^\infty(M)$, to be a linear map $D : C_p^\infty(M) \rightarrow \mathbb{R}$ such that

$$D(fg) = (Df)g(p) + f(p)Dg.$$

(Leibniz's Rule)

Note: The derivation is a map that takes functions into a real number. "It is similar as taking a derivative and evaluating it at a point."

Definition 1.3 (Tangent vector). A *tangent vector* at a point p in a manifold M is a derivation at p .

Just as for $\mathbb{R}^n$,

Definition 1.4 (Tangent Space). The tangent vectors at p form a vector space $T_p(M)$, called the tangent space of M at p .

1.3.3 Derivatives and Chain Rule

Definition 1.5 (Differential). Let $F : N \rightarrow M$ be a C^∞ map between two manifolds. At each point $p \in N$, the map F induces a linear map of tangent spaces, called its differential at p , $F_* : T_p N \rightarrow T_{F(p)} M$ as follows.

If $v_p \in T_p N$, then $F_*(v_p)$ is the tangent vector in $T_{F(p)} M$ defined by

$$(F_*(v_p))f = v_p(f \circ F) \in \mathbb{R} \quad \text{for } f \in C_{F(p)}^\infty(M)$$

Here f is a germ at $F(p)$, represented by a C^∞ function in a neighborhood of $F(p)$. Since (8.1) is independent of the representative of the germ, in practice we can be cavalier about the distinction between a germ and a representative function for the germ.

Definition 1.6 (Chain Rule). Let $F : N \rightarrow M$ and $G : M \rightarrow P$ be smooth maps of manifolds, and $p \in N$. The differentials of F at p and G at $F(p)$ are linear maps $T_p N \xrightarrow{F_*, p} T_{F(p)} M \xrightarrow{G_*, F(p)} T_{G(F(p))} P$.

If $F : N \rightarrow M$ and $G : M \rightarrow P$ are smooth maps of manifolds and $p \in N$, then

$$(G \circ F)_{*, p} = G_{*, F(p)} \circ F_{*, p}.$$

It is easier to visualize it in coordinates, where $F_{*, p}$ is simply the Jacobean matrix at point p .

1.3.4 Constant Rank Theorem

Theorem 1.7 (Constant Rank Theorem). *Let N and M be manifolds of dimensions n and m respectively. Suppose $f : N \rightarrow M$ has constant rank k in a neighborhood of a point p in N . Then there are charts (U, ϕ) centered at p in N and (V, ψ) centered at $f(p)$ in M such that for $(r_1, \dots, r_n)$ in $\phi(U)$,*

$$(\psi \circ f \circ \phi^{-1})(r_1, \dots, r_n) = (r_1, \dots, r_k, 0, \dots, 0).$$

Theorem 1.8 (Constant-Rank Level Set Theorem). *Let $f : N \rightarrow M$ be a C^∞ map of manifolds and $c \in M$. If f has constant rank k in a neighborhood of the level set $f^{-1}(c)$ in N , then $f^{-1}(c)$ is a regular submanifold of N of codimension k .*

Note: Here we can have that the constant rank $k \leq m$. It is also a local statement since the rank is required to be constant in a neighbourhood.

Proof. Let p be an arbitrary point in $f^{-1}(c)$. By the constant rank theorem there are a coordinate chart $(U, \phi) = (U, x_1, \dots, x_n)$ centered at $p \in N$ and a coordinate chart $(V, \psi) = (V, y_1, \dots, y_m)$ centered at $f(p) = c \in M$ such that

$$(\psi \circ f \circ \phi^{-1})(r_1, \dots, r_n) = (r_1, \dots, r_k, 0, \dots, 0) \in R^m.$$

This shows that the level set $(\psi \circ f \circ \phi^{-1})(0)$ is defined by the vanishing of the coordinates $r_1, \dots, r_k$.

$$\phi(f^{-1}(c)) = \phi(f^{-1}(\psi^{-1}(0))) = (\psi \circ f \circ \phi^{-1})^{-1}(0).$$

Thus, the level set $f^{-1}(c)$ in U is defined by the vanishing of the coordinate functions $x_1, \dots, x_k$, where $x_i = r_i \circ \phi$. This proves that $f^{-1}(c)$ is a regular submanifold of N of codimension k . $\square$

1.3.5 Inverse and Implicit Function Theorem

- **Inverse function theorem:** F is locally invertible if and only if its Jacobian determinant is not 0.
- **Implicit function theorem:** If dF is surjective, then $F(x, y) = c$ can be solved for y as a function of x , $y = h(x)$ and the solution set is locally the graph $\{x, h(x)\}$.

Theorem 1.9 (Inverse Function Theorem). *Let $F : N \rightarrow M$ be a C^∞ map between two manifolds of the same dimension, and $p \in N$. Suppose for some charts $(U, \phi) = (U, x_1, \dots, x_n)$ about p in N and $(V, \psi) = (V, y_1, \dots, y_n)$ about $F(p)$ in M , $F(U) \subset V$. Set $F_i = y_i \circ F$. Then F is locally invertible at p if and only if its Jacobian determinant $\det[\frac{\partial F_i}{\partial x_j}(p)]$ is nonzero.*

Theorem 1.10 (Implicit Function Theorem). *Let U be an open subset in $R^n \times R^m$ and $f : U \rightarrow R^m$ a C^∞ map. Write $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_m)$ for a point in U . At a point $(a, b) \in U$ where $f(a, b) = 0$ and the determinant $\det[\frac{\partial f_i}{\partial y_j}(a, b)]$ is nonzero, there exist a neighborhood $A \times B$ of (a, b) in U and a unique function $h : A \rightarrow B$ such that in $A \times B \subset U \subset R^n \times R^m$,*

$$f(x, y) = 0 \iff y = h(x).$$

Moreover, h is C^∞ .

1.3.6 Proof of Regular Value Theorem

There is a special case of the constant level-set theorem, which is the regular value theorem, where the differential map has the maximum possible rank, so $k = m$.

Theorem 1.11 (Regular Value Theorem). *Let $F : N \rightarrow M$ be a C^∞ map of manifolds, with $\dim(N) = n$ and $\dim(M) = m$. Then a nonempty regular level set $F^{-1}(c)$, where $c \in M$, is a regular sub-manifold of N of dimension equal to $n - m$.*

This theorem follows from the constant rank theorem, by allowing us to construct charts.

Proof. Let $F : N^n \rightarrow M^m$, $S = F^{-1}(c)$. The goal is to show that S is an $(n-m)$ -dimensional submanifold.

1. Turn the level set into a zero set (to apply later inverse function theorem)

Choose coordinates around $c \in M$:

$$\psi = (y^1, \dots, y^m), \quad \psi(c) = 0.$$

Then

$$F^{-1}(c) = (\psi \circ F)^{-1}(0).$$

Writing $F_i = y^i \circ F$, we get

$$S = \{F_1 = 0, \dots, F_m = 0\}.$$

So we have reduced the problem to understanding the common zero set of m functions.

2. Look at one point $p \in S$ Choose local coordinates $(x^1, \dots, x^n)$ around p .

Because c is a regular value, dF_p is surjective. Therefore

$$\text{rank}(dF_p) = m.$$

Equivalently, the Jacobian matrix

$$\left[\frac{\partial F_i}{\partial x^j}(p) \right]$$

has rank m .

Therefore, after possibly rearranging the coordinates, we can assume that the first $m \times m$ block

$$\left[\frac{\partial F_i}{\partial x^j}(p) \right]_{1 \leq i, j \leq m}$$

is invertible. This is the key use of regularity.

3. Use $F_1, \dots, F_m$ as new coordinates: Instead of using $(x^1, \dots, x^m, x^{m+1}, \dots, x^n)$, define new coordinates

$$(F_1, \dots, F_m, x^{m+1}, \dots, x^n).$$

Why are these actually coordinates?

Because the Jacobian of this coordinate change is

$$\begin{pmatrix} \frac{\partial F_i}{\partial x^j} & * \\ 0 & I \end{pmatrix}.$$

Its determinant is

$$\det \left[\frac{\partial F_i}{\partial x^j} \right]_{1 \leq i, j \leq m} \neq 0.$$

Therefore, by the Inverse Function Theorem, these functions really do give a coordinate system near p .

4. Describe the level set in the new coordinates: In these new coordinates, $S = F^{-1}(c)$ is simply

$$F_1 = \cdots = F_m = 0.$$

So locally it looks like

$$\{(0, \dots, 0, x^{m+1}, \dots, x^n)\}.$$

In other words,

$$S \cong \mathbb{R}^{n-m}$$

locally.

5. Conclude that S is a submanifold The new coordinate chart is an adapted chart for S . Since we can do this around every point $p \in S$, S is a regular submanifold of N . There are $n - m$ free coordinates, so $\dim S = n - m$

□

Proof. Suppose $F : N^n \rightarrow M^m$ and $c \in M$ is a regular value. Take $p \in F^{-1}(c)$. Because c is regular, dF_p is surjective, hence

$$\text{rank } dF_p = m.$$

By the constant rank theorem, locally we can choose coordinates so that

$$F(x_1, \dots, x_n) = (x_1, \dots, x_m).$$

Therefore

$$F^{-1}(c) = \{x_1 = \cdots = x_m = 0\}.$$

Hence

$$\boxed{F^{-1}(c) \text{ is locally } \mathbb{R}^{n-m}.$$

Since this works around every $p \in F^{-1}(c)$, $F^{-1}(c)$ is an $(n-m)$ -dimensional submanifold. □

1.3.7 Applications of Regular Value Theorem

Example 1.12 (Sphere). Define $F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$, $F(x) = |x|^2$. Then

$$S^n = F^{-1}(1).$$

Compute:

$$dF_x(v) = 2\langle x, v \rangle.$$

For $x \in S^n$, this map is surjective because $x \neq 0$. Therefore 1 is a regular value. Hence

$$\boxed{S^n \text{ is a smooth } n\text{-manifold.}}$$

And, importantly,

$$T_x S^n = \ker dF_x.$$

Thus

$$2\langle x, v \rangle = 0,$$

so

$$T_x S^n = \{v \in \mathbb{R}^{n+1} : \langle x, v \rangle = 0\}.$$

Draw the sphere and tangent plane.

Example 1.13 ($SL(n, \mathbb{R})$). (The special linear group, i.e. matrices whose determinant is 1) Consider $F : M_n(\mathbb{R}) \rightarrow \mathbb{R}$ such that $F(A) = \det A$. Then

$$SL(n, \mathbb{R}) = F^{-1}(1).$$

We need to show 1 is a regular value. The differential of determinant at A is

$$d(\det)_A(H) = \text{tr}(\text{adj}(A)H).$$

In particular, if $A \in SL(n, \mathbb{R})$, then A is invertible and

$$d(\det)_A(H) = \det(A) \text{tr}(A^{-1}H) = \text{tr}(A^{-1}H).$$

This is not the zero map. For example choose

$$H = A.$$

Then

$$d(\det)_A(A) = \text{tr}(I) = n \neq 0.$$

Thus $d(\det)_A$ is surjective.

Therefore $SL(n, \mathbb{R})$ is a smooth manifold of dimension $n^2 - 1$