Gromov-Hausdorff Metric Geometry

A proof of the generalized Borsuk problem

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Notation

Definition (Distance between points)

If X is a metric space, distance between $x, y \in X$ is written $|xy|$

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Definition (Diameter)

$$\operatorname{diam} X = \sup\{|xy| : x, y \in X\}$$

Hausdorff Distance

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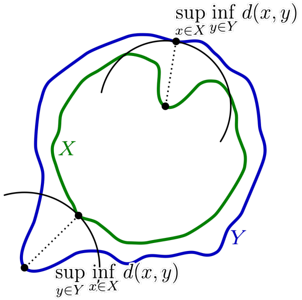

Gromov-Hausdorff Distance

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(X', Y', Z) is a realization of (X, Y), if $X', Y' \subset Z$ are isometric to X, Y.

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Theorem (6.4)

$$d_{GH}(X, Y) = \inf\{\rho_H(X, Y) : \rho \in \mathcal{D}(X, Y)\}$$

Distortions

Definition (Distortion)

Let $\sigma \in \mathcal{P}_0(X \times Y)$ (A non-empty relation)

$$\operatorname{dis}\sigma = \sup\{||xx'| - |yy'|| : (x, y), (x', y') \in \sigma\}$$

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Theorem (6.12)

$$d_{GH}(X, Y) = \frac{1}{2}\inf\{dis\,R : R \in \mathcal{R}(X, Y)\}$$

Theorem Statement

Theorem (8.7)

If X is a bounded metric space, $m < \#X$, and $\lambda < diam X$, then X can be partitioned in m subsets with strictly smaller diameters iff $2d_{GH}(\lambda\Delta_m, X) < diam X$

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Definition (Simplexes)

A simplex is a metric space where all non-zero distances are equal. $\lambda\Delta_m$ is the simplex with cardinality m and non-zero distance λ

Distances to Simplexes

Theorem (8.5)

If X is bounded and $m \leq \#X$, then

$$\begin{aligned} 2d_{GH}(\lambda\Delta_m, X) &= \inf_{D \in \mathcal{D}_m(X)} \max\{diamD, \lambda - \alpha(D), diamX - \lambda\} \\ &\leq diamX \end{aligned}$$

Where $\mathcal{D}_m(X)$ is the set of partitions into m subsets and $diamD = \sup\{diamX : X \in D\}$

Proof

Theorem (8.7)

If X is a bounded metric space, $m < \#X$, and $\lambda < diamX$, then X can be partitioned in m subsets with strictly smaller diameters iff $2d_{GH}(\lambda\Delta_m, X) < diamX$

Proof

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If X is a bounded metric space, $m < \#X$, and $\lambda < diam X$, then X can be partitioned in m subsets with strictly smaller diameters iff $2d_{GH}(\lambda\Delta_m, X) < diam X$

- $2d_{GH}(\lambda\Delta_m, X) \leq \text{diam} X$ by 8.5
- Equality holds iff $\text{diam} D = \text{diam} X$ which only happens when there are no partitions with smaller diameters

Sources

http://dfgm.math.msu.su/people/tuzhilin/English/AlexeyTuzhilin.html
https://arxiv.org/abs/2012.00756