

Integration Seminar by Lilly Montelle (10/5/26)

Topics

- 1) Orientation and Forms
- 2) Integration on Manifolds
- 3) Stokes' Theorem

References

Tu W., Loring. An Introduction to Manifolds. Second Edition. (2010). Springer. New York, NY.

Guillemin, Victor and Pollack, Alon. Differential Topology. (1974). Prentice-Hall, Inc. Englewood Cliffs, New Jersey.

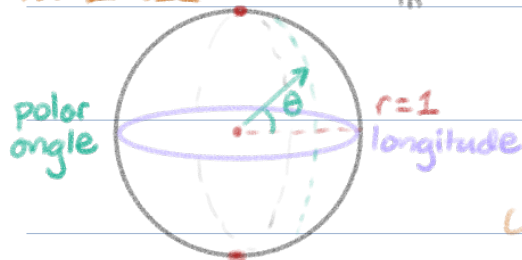
Arnold, V.I. Mathematical Methods of Classical Mechanics. Second Edition. (1989). Springer. New York, NY.

Propose a Function on S^2 to integrate

Tu 23.11

$\mathbb{R}^3$

↳ closed 2-D manifold with no boundary



Sphere S^2 has a parametrization by spherical

coordinates $F(\phi, \theta) = (\sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi)$ on

$\underbrace{\text{polar angle}}_{\theta} \underbrace{\text{longitude}}_{\phi}$

$$\omega = \sin(\phi) d\phi \wedge d\theta$$

$$D = \{(\phi, \theta) \in \mathbb{R}^2 \mid 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$$

Standard Volume Form from S^2 in certain coordinates

Hairy Ball Theorem: you cannot comb a hairy ball flat without

creating a cowlick → cannot comb a smooth vector field on an

even-dimensional sphere without containing ≥ 1 where the vector = 0

We can integrate differential forms by defining integration

Stokes' Theorem: Let M be an oriented manifold of dimension

n with boundary ∂M & given boundary orientation. Let

$i: \partial M \hookrightarrow M$ be the inclusion map. For any smooth $(n-1)$ -

form ω with compact support on the oriented n -dimensional

manifold M , $\int_M d\omega = \int_{\partial M} \omega.$

To prove, we must establish three prior components

Let $\omega = dx^1 \wedge \dots \wedge dx^n$ be the volume form on $\mathbb{R}^n$

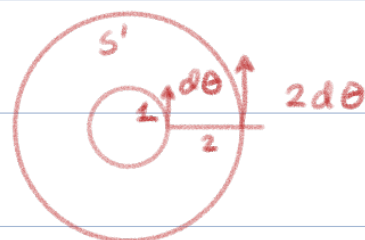
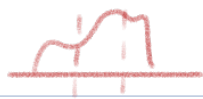
① Volume Forms: determine which form on a patch corresponds to the usual integration, allowing us to integrate forms in terms of Riemann integrals. (dV_g)

a) There is an n -form on an n -dimensional manifold in bijection with functions on the manifold

b) You can always pick on a manifold

$$n\text{-form on } M^n \Leftrightarrow C^\infty(M)$$

Ex) $\mathbb{R}^1$



$\mathbb{R}^3$ length of the interval summed by the weights of intervals in the domain $\Rightarrow$ scale linearly

Ex) $\int f(x,y) dx dy \Rightarrow \int f(x,y)$ on Euclidean space, or you could say $\Rightarrow \int f(x,y) e^x dx dy$ is its correspondence

The choice of volume form defines correspondence

If you integrate the constant function 1 against the volume form, you get the volume of a manifold. Compare this to the integral on a closed interval, providing length.

$$dV_g = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2$$

where (n_1, n_2, n_3) is the unit normal vector

Orientable if the smooth coordinate neighborhood of a point p is diffeomorphic to an open subset of Euclidean space

Def: Suppose (M, g) is an oriented Riemannian n -manifold with or w/o boundary, and $n \geq 1$. $\exists!$ smooth orientation form $\omega_g \in \Omega^n(M)$, denoted the Riemannian volume form, satisfying $\omega_g(E_1, \dots, E_n) = 1$ for every local oriented orthonormal frame (E_i) for M .

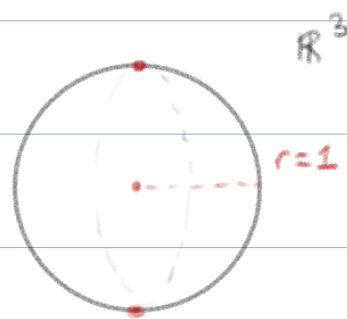
Applying to our function on S^2

① Volume Form

$\omega = x dy \wedge dz - y dx \wedge dz + z dx \wedge dy$, then apply the

spherical coordinate transformation $x = \sin \phi \cos \theta$, $y = \sin \phi \sin \theta$,

and $z = \cos \phi$



$$\omega = \sin(\phi) d\phi \wedge d\theta$$

②

Pullbacks: let us integrate on a manifold by integrating on Euclidean space

→ Pullback of h by F

Let $F: N \rightarrow M$ be a map and h a function on M . F^*h is the composite function $h \circ F$.

Ex)  $M \times \mathbb{R}^n (x, F(x)) \Leftrightarrow x \mapsto F(x)$ $\mathbb{R}^n(M) \ni M$

a) Choices matter - value depends on the Jacobian determinant

b) This always exists if there is a globally-defined function, and a non-zero one always exists

c) Every open interval is diffeomorphic in $\mathbb{R}$, so pick an infinitesimal notion of length depending on the choice of volume

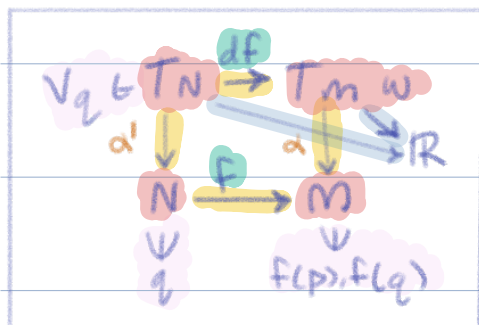
d) On a chart, pullback to $\mathbb{R}^n \Rightarrow \det(\text{derivative})$

Ex) One-form

$$\left. \begin{array}{l} V \xrightarrow{F} W \xrightarrow{g} Z \\ \downarrow \quad \downarrow \\ g \circ F \end{array} \right\} \text{differential form on a tangent bundle}$$

Apply df to each n -input

$df(V_q) \in T_{F(q)} M^W \rightarrow \mathbb{R}$ $f: M \rightarrow \mathbb{R} \rightsquigarrow T_m \rightarrow \mathbb{R}$ $W_p: T_p M \rightarrow \mathbb{R}$



$(F^*w_p)^{(q)} = w_{F(q)}(df(v_q))$

$w \in \Omega^1(M)$ $f: N \rightarrow M \rightsquigarrow f^*w \in \Omega^1(N)$

$w \in \Omega^{top}(M)$ $f^*w \in \Omega^{top}(N)$

$(\det df) w_{vol(N)}$

$w \in \Omega^*(N)$ $w|_U \in \Omega^*(U)$

$\mu^*w \in \Omega(\mu(N))$

Commutative Rings

$\mu \downarrow \mathbb{R}^n$

$\hookrightarrow \int \mu^*w$

$d \circ df = F \circ \alpha'$

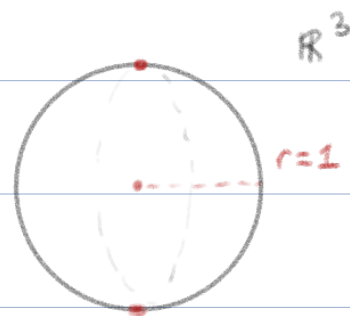
If $f: X \rightarrow Y$ is any smooth map of two k -dimensional manifolds with boundary, the pullback f^*V_Y is a k -form on X . At every point $x \in X$, $V_x(x)$ is a basis for $\Delta^k[T_x(x)^*]$, so $(f^*V_Y)(x)$ must be a scalar multiple of $V_x(x)$. **Jacobian.**

We show the 2-form defined by $(v, w) \mapsto \frac{1}{2} \det \begin{pmatrix} v \\ w \end{pmatrix}$ for pairs of vectors v and w in $\mathbb{R}^3$

Applying to our function on S^2

② Pullback (Using Tu's computation)

$S^2 \rightarrow \mathbb{R}^3$ embedding map



$$F^*dx = \cos\phi \cos\theta d\phi - \sin\phi \sin\theta d\theta$$

$$F^*dy = \cos\phi \sin\theta d\phi + \sin\phi \cos\theta d\theta$$

$$F^*dz = -\sin\phi d\phi$$

$$\omega = \sin(\phi) d\phi \wedge d\theta$$

Compute Wedge Products

$$dy \wedge dz = (\cos\phi \sin\theta d\phi + \sin\phi \cos\theta d\theta) \wedge (-\sin\phi d\phi) = \sin^2\phi \cos\theta (d\phi \wedge d\theta)$$

$$dx \wedge dz = (\cos\phi \cos\theta d\phi - \sin\phi \sin\theta d\theta) \wedge (-\sin\phi d\phi) = -\sin^2\phi \sin\theta (d\phi \wedge d\theta)$$

$$dx \wedge dy = (\cos\phi \cos\theta d\phi - \sin\phi \sin\theta d\theta) \wedge (\cos\phi \sin\theta d\phi + \sin\phi \cos\theta d\theta) = \sin\phi \cos\phi (d\phi \wedge d\theta)$$

Substitute

$$\begin{aligned} \omega = & \underbrace{(\sin\phi \cos\theta)}_x \underbrace{[\sin^2\phi \cos\theta (d\phi \wedge d\theta)]}_{dy \wedge dz} - \underbrace{(\sin\phi \sin\theta)}_y \underbrace{[-\sin^2\phi \sin\theta (d\phi \wedge d\theta)]}_{dx \wedge dz} \\ & + \underbrace{(\cos\phi)}_z \underbrace{[-\sin\phi \cos\phi (d\phi \wedge d\theta)]}_{dx \wedge dy} \Rightarrow \sin(\phi) d\phi \wedge d\theta \end{aligned}$$

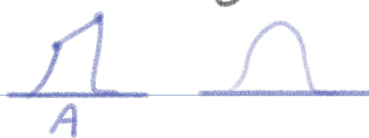
③

Partitions of Unity: allows us to patch local

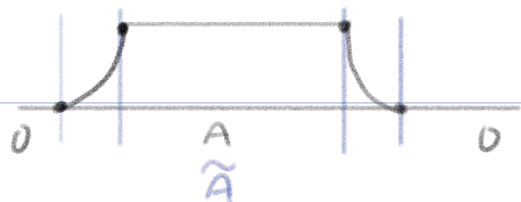
integrals together into a global one on a manifold



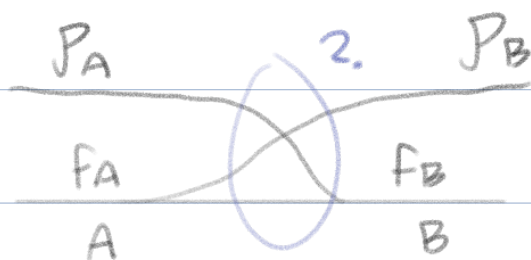
$F: A \rightarrow \mathbb{R}$



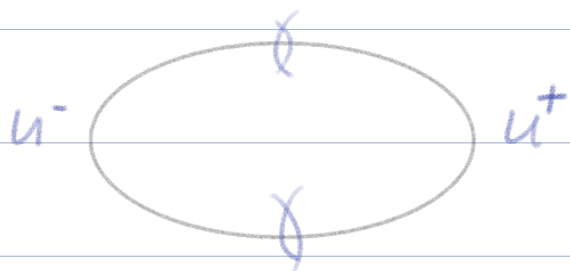
Collection of non-negative Functions summing to one



$P_A F = F \chi_A = 0$ outside of $A: \mathbb{R} \rightarrow \mathbb{R}$



$P_A F_A + P_B F_B$ but $P_A + P_B = 1$



$f: S^1 \rightarrow \mathbb{R}$

$P_+ P_-$

$f^+: U^+ \rightarrow \mathbb{R}$

$f^-: U^- \rightarrow \mathbb{R}$

$$\underbrace{\int P_+ f_+}_{U^+} + \underbrace{\int P_- f_-}_{U^-} = f = \int f$$

we work with smooth Functions & don't require Taylor Series

°] smooth functions s.t. not analytic $\rightarrow$ you can take infinite derivatives but the derived Taylor series does not agree

° Trivial Vector Bundle $\rightarrow$ section as a graph

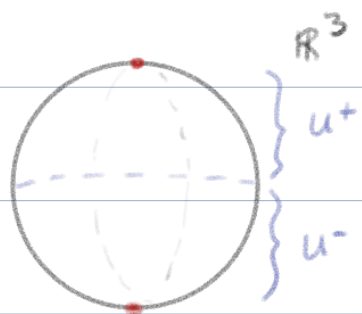
Choice used to define the integral on the whole manifold doesn't matter $\rightarrow$ yields some result

Thm: Existence of Partitions of Unity

Suppose M is a smooth manifold with or w/o boundary, and $X = (X_\alpha)_{\alpha \in A}$ is any indexed open cover of M . Then $\exists$ a smooth partition of unity subordinate to X .

Applying to our function on S^2

③ Partitions of Unity



$$w = \sin(\phi) d\phi \wedge d\theta$$

Split into two closed & smooth forms

Account for undefined north & south poles

No such global w , if it were, $\int_{S^2} \partial w = \int_{\partial S^2} w$

$\hookrightarrow$ boundary of S^2 is empty $\Rightarrow$ integral is zero $\times$

$\sin = 0$

Undefined at the connection line where $\phi = 0, \pm\pi, \pm 2\pi$

But in calculus if you change the value at a single point it doesn't change the integral. In 2-D if you change the value of a function along a line, the integral won't change.

Tu 23.11 tells us that the globally defined form agrees with our Ω except on the poles. Thus we can unravel the sphere into a rectangle away from it.

Local Potentials: $d(-\cos\phi d\theta) = \sin\phi d\phi \wedge d\theta \rightarrow \alpha = (1 - \cos\phi) d\theta$

$\nearrow$ chosen at the poles

$$\alpha_+ = (1 - \cos(\frac{\pi}{2})) d\theta = d\theta$$

$$\alpha_- = (-1 - \cos(\frac{\pi}{2})) d\theta = -d\theta$$

$$\int_{u^+} \omega = \int_{du^+} d\omega = \int_{S^1} d\omega$$

$$\int_{u^-} \omega = \int_{du^-} d\omega = \int_{-S^1} d\omega = - \int_{S^1} d\omega$$

From there, apply Stoke's Theorem and compute along the boundary.

Now we can apply Stokes' Theorem $\leadsto$ boundary on a manifold, locally homeo to $\mathbb{R}^n > 0$ on an $x-1$ axis

1) Assume integration on $\mathbb{R}^n$

*Fubini's Theorem

2) Fix the orientation

3) State the change of volume formula for S'

4) Conduct pullback

$$\int_M dw = \int_{\partial M} w \quad f(x) \rightarrow \frac{\partial f}{\partial x} dx$$

$$\partial w \in \Omega^1(M) \quad w \in \Omega^*(M) = C^\infty(M)$$

$$M = [a, b] \subseteq \mathbb{R}$$

$$\int_{[a, b]} \frac{\partial f}{\partial x} dx = \int_{\{a\}^+ \cup \{b\}^-} f(x)$$

$$\downarrow$$

$$\partial M = \{a\}^+ \cup \{b\}^- \text{ on a manifold}$$

$$\int_a^b \frac{\partial f}{\partial x}(x) dx = f(b) - f(a) \rightarrow \text{Fundamental theorem of Calculus}$$

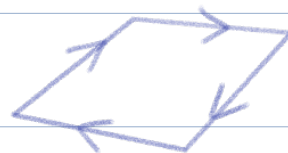
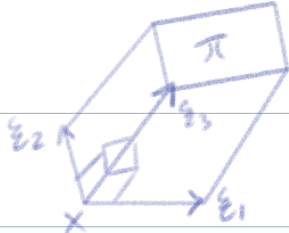


$$O S^2$$

$$\int_{S^2} dw = \int_{\emptyset} w = 0$$

From here we derive additional notions like Green's and the Curl and Divergence theorems!

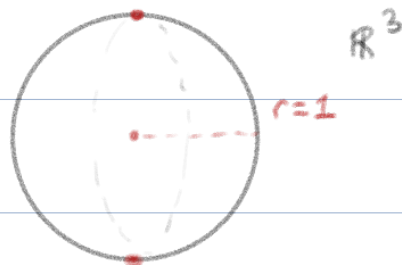
Note: Arnold



$$\lim_{n \rightarrow 0} \left(\frac{f(x+n) - f(x)}{n} \right)$$

$$\lim_{n \rightarrow 0} \left(\frac{\int_{[x+n, x]} f(x)}{\int_{[x+n, x]} \text{vol}} \right) \Rightarrow \lim_{| \Delta S | \rightarrow 0} \left(\frac{\int_{n\text{-cube}} w}{\int_{n\text{-cube}} \text{vol}} \right) = df$$

Applying to our function on S^2



$$w = \sin(\phi) d\phi d\theta$$

$$\int_{u^+} dw = \int_{\partial u^+} w \quad \text{and} \quad \int_{u^-} dw = \int_{\partial u^-} w \quad \int_{S^2} dw = \int_{u^+} dw + \int_{u^-} dw = \int_{\partial u^+} w + \int_{\partial u^-} w$$

$$\int_{S^2} w = \int_{S^1} (d\theta - (-d\theta)) = \int_0^{2\pi} 2d\theta = 4\pi$$

Appendix

①

Orientation on a Manifold:

a) Choice of an orientation for each tangent space satisfying a continuity condition

b) Globalizing n -covectors over a manifold derives differential n -Forms (on an n -manifold)

i) Equivalence class of C^∞ nowhere-vanishing n -Forms

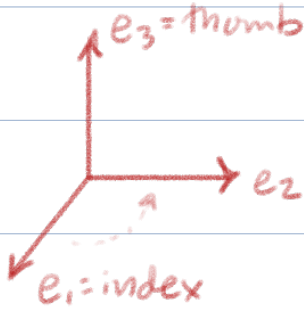
ii) Equivalent iff one is a multiple of the other via a positive function

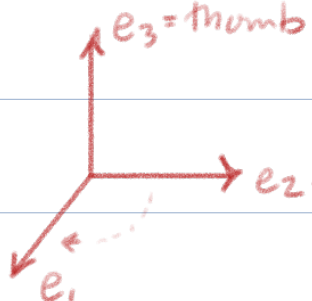
c) Oriented Atlas: atlas s.t. any two overlapping charts are related by a transition function with everywhere positive Jacobian determinant

Ex: $\mathbb{R}^0$: w/o a basis, thus + and - on $\{0\}$

$\mathbb{R}^1$  line orientations

$\mathbb{R}^2$  plane orientations

$\mathbb{R}^3$ 
Right-handed


Left-handed

$\mathbb{R}^n$: Orientation specified by an ordered basis

Let $e_1, \dots, e_n$ be S.B. for $\mathbb{R}^n$. For any two ordered basis (u_1, u_2) and (v_1, v_2) for $\mathbb{R}^2$, $\exists$ non-singular 2×2 matrix $A = [a_{ij}]$ where $u_j = \sum_{i=1}^2 v_i a_{ij}$, $j = 1, 2$

$\begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} i \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} \sum_{i=1}^2 v_i a_{i1} & \sum_{i=1}^2 v_i a_{i2} \\ \sum_{i=1}^2 v_i a_{i1}^2 & \sum_{i=1}^2 v_i a_{i2}^2 \end{bmatrix} \end{matrix}$ $\mathbb{R}^2$ This is the Change-of-Basis matrix
From (v_1, v_2) to (u_1, u_2)

i) Two order basis equivalent if the matrix A has a positive determinant

ii) Each equivalence class is an orientation of $\mathbb{R}^2$

Now let's generalize

- i) Assume all vector spaces finite-dimensional
- ii) Two ordered basis $u = [u_1, \dots, u_n]$ and $v = [v_1, \dots, v_n]$ of vector space V are said to be equivalent if $u = vA$ for an $n \times n$ matrix A with positive determinant ($u \sim v$)
- iii) Orientation of V is an equivalence class of ordered basis
- iv) Any finite-dimensional vector space has two orientations

↳ if μ is an orientation of finite-dimensional vector space V , denote other by $-\mu$ as "opposite" of the orientation μ

Notation

- a) Basis: $v_1, \dots, v_n$
- b) Ordered Basis: $(v_1, \dots, v_n)$
- c) Matrix Notation via Row Vector: $[v_1, \dots, v_n]$
- d) Orientation: $[(v_1, \dots, v_n)]$  equivalence class

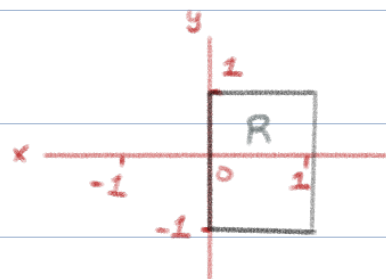
Ex: The Open Mobius Band

Let R be the rectangle $R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -1 \leq y \leq 1\}$.

The open Mobius band M is the quotient of R by the equivalence relation generated by $(0, y) \sim (1, -y)$.

The interior of R is the open rectangle

$$U = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1, -1 < y < 1\}$$



Suppose M is orientable. An orientation on M restricts to an orientation on U . For definiteness, assume orientation given by e_1, e_2 . By continuity, orientations at points $(0, 0)$ and $(1, 0)$ are also given by e_1, e_2 . Under definition of n -Covectors, ordered basis e_1, e_2 at $(1, 0)$ maps to $e_1, -e_2$ at $(0, 0)$. Thus at $(0, 0)$ the orientation is given by both e_1, e_2 and $e_1, -e_2$ (a contradiction).

Therefore M is not orientable.

↳ assuming e_2, e_1 orientation also leads to a contradiction

□

Value Forms (Arnold & Guillemin, Pollock)

Provides a means of integrating functions. If g a function on X , then $g \omega_x$ is a k -form on X . Thus define $\int_X g$ as $\int_X g \omega_x$

↳ When X is $\mathbb{R}^k$, $\omega_x = dx_1 \wedge \dots \wedge dx_k$ s.t. $\int_{\mathbb{R}^k} g$ is the usual integral

a) Poisson structure classification on a plane may be considered classification of differential forms of type $f(dx \wedge dy)^{-1}$ where f is a smooth function

↳ consider forms of type $f(dx)^\alpha = f(x_1, \dots, x_n)(dx_1 \wedge \dots \wedge dx_n)^\alpha$ where α is a fixed number (generally complex)

b) This classification and deformations in one-dimension revealed the role of resonance values of $\alpha \rightarrow$ ^{certain negative} rational numbers

Ex: resonance case $n=1, \alpha=-1$ corresponds to the classification of the singularities & their bifurcations for vector fields on the line (i.e. singular points of differential equations $\dot{x} = v(x)$ and their bifurcations in finite-parameter families)

② Integration on Manifolds:

a) Integral of an n -form w/ compact support in an open set of $\mathbb{R}^n$ is defined as the Riemann integral of the corresponding function

b) Closed rectangle in $\mathbb{R}^n$ is a Cartesian product $R = [a^1, b^1] \times \dots \times [a^n, b^n]$ of closed intervals in $\mathbb{R}$ where $a^i, b^i \in \mathbb{R}$

c) Let $f: R \rightarrow \mathbb{R}$ be a bounded function defined on a closed rectangle R , $\text{Vol}(R) := \prod_{i=1}^n (b_i - a_i)$

d) Partition of $[a, b]$ is $\{p_0, \dots, p_n\} \in \mathbb{R}$ s.t.

$$a = p_0 < p_1 < \dots < p_n = b$$

i) of R is collection $P = \{P_1, \dots, P_n\}$ where each P_i is a partition of $[a^i, b^i]$

$\hookrightarrow$ divides R into closed subrectangles R_j

$$\text{ii) } L(f, P) := \sum_{R_j} (\inf_{R_j} f) \text{vol}(R_j) \quad \text{lower sum}$$

$$U(f, P) := \sum_{R_j} (\sup_{R_j} f) \text{vol}(R_j) \quad \text{upper sum}$$

where each sum runs over all subrectangles of P

e) For any partition P , $L(f, P) \leq U(f, P)$ and for any two partitions P and P' of rectangle R , $L(f, P) \leq U(f, P')$

• Generalizing $\mathbb{R}^n$ integration to n -Forms on $\mathbb{R}^n$

a) Let $w = f(x) dx' \wedge \dots \wedge dx''$ be a C^∞ n -form on an open subset $U \subset \mathbb{R}^n$ w/ standard coordinates $x', \dots, x''$. Its integral over subset $A \subset U$ is defined as the Riemann integral of $f(x)$ if it exists:

$$\int_A w = \int_A f(x) dx' \wedge \dots \wedge dx'' := \int_A f(x) dx' \dots dx''$$

b) n -Form written in the order $dx' \wedge \dots \wedge dx''$

Ex: $\tau = f(x) dx^2 \wedge dx^1$ over $A \subset \mathbb{R}^2$

$$\int_A \tau = \int_A -f(x) dx^1 \wedge dx^2 = - \int_A f(x) dx^1 dx^2$$

• Integral of a Differential Form over a Manifold

a) Manifold must be oriented ($\mathbb{R}^n$ has a standard orientation)

b) On manifold of dimension n , one can integrate only n -forms, not functions

c) n -forms must have compact support

Let w be an n -form with compact support on an oriented manifold M of dimension n . If $-M$ denotes the same manifold of opposite orientation, $\int_{-M} w = - \int_M w$

↳ Reversing orientation of M reverses the sign of an integral over M

Zero-Dimensional Manifold $\rightarrow$ compact oriented manifold

M of $\dim=0$ is a finite collection of points oriented by $+1$ or -1

as $M = \sum p_i - \sum q_j$. Integral of 0-form $f: M \rightarrow \mathbb{R}$ defined as

$$\int_M f = \sum f(p_i) - \sum f(q_j)$$

③ Stokes' Theorem:

Assumptions: Let M be an oriented manifold of dimension n with boundary ∂M & boundary ^{given} orientation. Let $i: \partial M \hookrightarrow M$ be the inclusion map. If w is $(n-1)$ -form on M , write $\int_{\partial M} w$ instead of $\int_{\partial M} i^* w$.

For any smooth $(n-1)$ -form w with compact support on the oriented n -dimensional manifold M , $\int_M dw = \int_{\partial M} w$.

Proof

a) Choose atlas $\{(U_\alpha, \phi_\alpha)\}$ for M s.t. each U_α is diffeomorphic to $\mathbb{R}^n$ or $\mathbb{H}^n$ via orientation-preservation
 $\hookrightarrow$ any open disk is diffeomorphic to $\mathbb{R}^n$ & any half-disk containing its boundary diameter is diffeomorphic to $\mathbb{H}^n$

b) Let $\{p_\alpha\}$ be a C^∞ partition of unity subordinate to $\{U_\alpha\}$. Shown prior, $(n-1)$ -form $p_\alpha w$ has compact support in U_α .

c) Suppose this holds for $\mathbb{R}^n$ and $\mathbb{H}^n$. Then it holds for all charts U_α in our atlas, which are diffeomorphic to $\mathbb{R}^n$ or $\mathbb{H}^n$

d) Observe that $(\partial M) \cap U_\alpha = \partial U_\alpha$, and thus

$$\begin{aligned}
 \int_{\partial M} w &= \int_{\partial M} \sum_{\alpha} p_{\alpha} w && \left(\sum_{\alpha} p_{\alpha} = 1 \right) \rightarrow \text{proven prior} \\
 &= \sum_{\alpha} \int_{\partial M} p_{\alpha} w && \left(\sum_{\alpha} p_{\alpha} w \text{ is a finite sum} \right) \\
 &= \sum_{\alpha} \int_{\partial U_{\alpha}} p_{\alpha} w && \left(\text{suppose } p_{\alpha} w \subset U_{\alpha} \right) \\
 &= \sum_{\alpha} \int_{U_{\alpha}} d(p_{\alpha} w) && \left(\text{Stokes' Theorem for } U_{\alpha} \right) \\
 &= \sum_{\alpha} \int_M d(p_{\alpha} w) && \left(\text{suppose } d(p_{\alpha} w) \subset U_{\alpha} \right) \\
 &= \int_M d\left(\sum_{\alpha} p_{\alpha} w\right) && \left(\sum_{\alpha} p_{\alpha} w \text{ is a finite sum} \right) \\
 &= \int_M dw
 \end{aligned}$$

□

Thus it suffices to prove the theorem for $\mathbb{R}^n$ and $\mathbb{H}^n$

↳ $\mathbb{H}^2$ proof is highly similar to $\mathbb{H}^n$

Proof for the Upper half-plane $\mathbb{H}^2$

a) Let x, y be coordinates on $\mathbb{H}^2$, then standard orientation given by $dx \wedge dy$ and boundary orientation on $\partial\mathbb{H}^2$ given by $2 - \partial/\partial y (dx \wedge dy) = dx$. Form w is a linear combination $w = f(x, y)dx + g(x, y)dy$ for C^∞ Functions f, g with compact support in $\mathbb{H}^2$

b) Supports of f, g compact and thus choose $a > 0, a \in \mathbb{R}$ large enough s.t. supports of f, g contained in square interior $[-a, a] \times [0, a]$. Use F_x, F_y to denote partial derivatives of f w/ respect to x and y .

c) Then $dw = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy = (g_x - f_y) dx \wedge dy$ and

$$\int_{\mathbb{H}^2} dw = \int_{\mathbb{H}^2} g_x dx dy - \int_{\mathbb{H}^2} f_y dx dy$$

$$= \int_0^\infty \int_{-\infty}^\infty g_x dx dy - \int_{-\infty}^\infty \int_0^\infty f_y dy dx$$

$$= \int_0^a \int_{-a}^a g_x dx dy - \int_{-a}^a \int_0^a f_y dy dx \quad *$$

$$\int_{-a}^a g_x(x, y) dx = g(x, y) \Big|_{x=-a}^a = 0 \text{ b/c suppose } g \text{ in interior of } [-a, a] \times [0, a]$$

d) Similarly, $\int_0^a f_y(x, y) dy = f(x, y) \Big|_{y=0}^a = -f(x, 0)$ b/c $f(x, a) = 0$

e) Thus $\star$ becomes $\int_{H^2} dw = \int_{-a}^a f(x, 0) dx$

f) ∂H^2 is the x-axis and $dy = 0$ on ∂H^2 , so $w = f(x, 0) dx$ when restricted to ∂H^2 and $\int_{\partial H^2} w = \int_{-a}^a f(x, 0) dx$

□

Fubini's Theorem

If a function is Lebesgue integrable on $X \times Y$ rectangle,

$$\iint_{X \times Y} f(x, y) d(x, y) = \int_X \left(\int_Y f(x, y) dy \right) dx = \int_Y \left(\int_X f(x, y) dx \right) dy$$