

Marked in red: extra information
that I will say
Blue: Main info

TALK

1. Motivation
2. Inner products on vector space
3. Riemann metrics + examples
4. some implications

1. Motivation.

Arc length of graph

on $[a, b]$: for $r(t) = (x(t), y(t))$

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt = \int_a^b |r'(t)| dt, \quad |v| = \sqrt{v \cdot v}$$

only geometry inside this formula is dot product,
speed is $\sqrt{r' \cdot r'}$ but you don't know how

long a vector is on a manifold:

— velocities are just tangent vectors, tangent
space is vector space (scale and add) but no lengths.

→ need to choose rule to give us lengths

2. Inner products.

Def. inner product on vector space V is $\langle, \rangle: V \times V \rightarrow \mathbb{R}$
which is

$$w, u, v \in V \\ a \in \mathbb{R}$$

i. bilinear: $\langle au, v \rangle = a \langle u, v \rangle$.

$$\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$

and vice versa

ii. symmetric: $\langle u, v \rangle = \langle v, u \rangle$

iii. positive definite: $\langle v, v \rangle \geq 0$, with equality

if and only if $v = 0$

as p varies, g_p varies smoothly, entries of $g_{ij}(p)$ smooth

Def. Riemannian metric is a smoothly varying inner product at tangent space at each point

$$g_p: T_p M \times T_p M \rightarrow \mathbb{R}, \quad p \in M$$

(g is metric tensor) - inner product at p .
takes pairs of vectors and spits number

Riemannian manifold is differentiable manifold with
Riemannian metric g .

$$T_p M \cong V \quad \text{where } \alpha_1, \alpha_2 \text{ are basis (at } p \text{)}$$

here $x = x_1 \alpha_1 + x_2 \alpha_2$, $y = y_1 \alpha_1 + y_2 \alpha_2$

try to find $\langle x, y \rangle$. Assuming bilinearity, symmetry

$$\langle x_1 \alpha_1 + x_2 \alpha_2, y_1 \alpha_1 + y_2 \alpha_2 \rangle$$

$$= \langle x_1 \alpha_1, y_1 \alpha_1 + y_2 \alpha_2 \rangle + \langle x_2 \alpha_2, y_1 \alpha_1 + y_2 \alpha_2 \rangle$$

$$= x_1 \langle \alpha_1, y_1 \alpha_1 + y_2 \alpha_2 \rangle + x_2 \langle \alpha_2, y_1 \alpha_1 + y_2 \alpha_2 \rangle$$

$$\begin{aligned}
&= x_1 \langle \alpha_1, y_1 \alpha_1 \rangle + x_1 \langle \alpha_1, y_2 \alpha_2 \rangle \\
&\quad + x_2 \langle \alpha_2, y_1 \alpha_1 \rangle + x_2 \langle \alpha_2, y_2 \alpha_2 \rangle \\
&= x_1 y_1 \langle \alpha_1, \alpha_1 \rangle + x_1 y_2 \langle \alpha_1, \alpha_2 \rangle \\
&\quad + x_2 y_1 \langle \alpha_2, \alpha_1 \rangle + x_2 y_2 \langle \alpha_2, \alpha_2 \rangle \\
&= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \langle \alpha_1, \alpha_1 \rangle & \langle \alpha_1, \alpha_2 \rangle \\ \langle \alpha_2, \alpha_1 \rangle & \langle \alpha_2, \alpha_2 \rangle \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
\end{aligned}$$

(The matrix in the middle is circled and labeled G_{CP})

$$= \langle x, y \rangle \quad \text{or can also denote as } g_p(x, y) = x^T G y$$

Notice positive definite G is symmetric because $\langle \alpha_1, \alpha_2 \rangle = \langle \alpha_2, \alpha_1 \rangle$
 because $\langle x, y \rangle$ is pos def, by definition $x^T G y = \langle x, y \rangle$ is pos def. Apdx 2.

define G on basis elements as follows: α_1, α_2

$$G_{11} = \langle \alpha_1, \alpha_1 \rangle \quad G_{12} = \langle \alpha_1, \alpha_2 \rangle$$

$$G_{21} = \langle \alpha_2, \alpha_1 \rangle \quad G_{22} = \langle \alpha_2, \alpha_2 \rangle$$

can represent this as $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$

— Capital G is matrix that represents g (not a matrix, just mapping)

— G smoothly varies $\rightarrow$ each entry is smooth function
 — smoothly varying is local property

Example: Ordinary plane $M = \mathbb{R}^2$, taking dot product on all tangent space in standard basis: $\rightarrow$ same old nonvarying in $\mathbb{R}^2$, global, uniform

$e_1 = \partial_x \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $e_2 = \partial_y \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

dotted $g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

which is a constant factor $\rightarrow$ smooth.
 $\rightarrow$ this is metric on standard basis of Euclidean space, so it is the standard Euclidean metric

how do we get the arclength form? we want to get the length of a vector, which is $\sqrt{\langle v, v \rangle} = \|v\|$

for vector $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ we have $dx(v) = v_1$, $dy(v) = v_2$

$\langle v, v \rangle = g_{11} dx^2 + 2g_{12} dx dy + g_{22} dy^2 \rightarrow g$
 $[= g_{11} v_1^2 + 2g_{12} v_1 v_2 + g_{22} v_2^2] \rightarrow g(v, v)$

but for standard Euclidean metric $g_{11}, g_{22} = 1$, $g_{12} = 0$

so we have $v_1^2 + v_2^2 = dx^2 + dy^2$ (quadratic form) which is length element $(dx^2 + dy^2)^{1/2}(v)$ to be precise

(if you polarize this you get back Euclidean metric, which is what some call it - same info)

with tiny step $S(ds^2 = dx^2 + dy^2)$ over tiny t ,

$$\left(\frac{ds}{dt}\right)^2 = g_{11} \left(\frac{dx}{dt}\right)^2 + 2g_{12} \frac{dx}{dt} \frac{dy}{dt} + g_{22} \left(\frac{dy}{dt}\right)^2$$

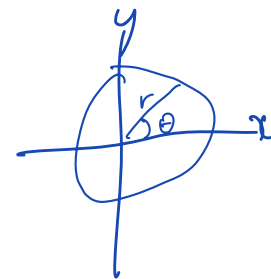
write in derivatives:

$$s(t) = \int_a^t |r'(\tau)| d\tau = \int \sqrt{g_{\text{ref}}(r(t), r'(t))} dt$$

$$L(r) = \int_a^b \sqrt{g(r'(t), r'(t))} dt, \text{ using } r(t) = (x(t), y(t))$$

$$L = \int_a^b \sqrt{g_{11}\dot{x}^2 + 2g_{12}\dot{x}\dot{y} + g_{22}\dot{y}^2} dt \rightarrow \int \sqrt{\dot{x}^2 + \dot{y}^2} \quad (\text{traditional physics notation})$$

3 Examples S^1



$$\varphi: D \xrightarrow{\text{domain}} [0, 2\pi) \times \mathbb{R}_+ \rightarrow \mathbb{R}^2 \setminus \{0\}$$

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\varphi^* (dx^2 + dy^2)$$

$$= d(r \cos \theta)^2 + d(r \sin \theta)^2$$

$$= (\cos \theta dr - r \sin \theta d\theta)^2 + (\sin \theta dr + r \cos \theta d\theta)^2$$

$$= \cos^2 \theta dr^2 - 2r \sin \theta \cos \theta dr d\theta + r^2 \sin^2 \theta d\theta^2 + \sin^2 \theta dr^2 + 2r \sin \theta \cos \theta dr d\theta + r^2 \cos^2 \theta d\theta^2$$

$$= r^2 (\sin^2 \theta + \cos^2 \theta) d\theta^2 + (\sin^2 \theta + \cos^2 \theta) dr^2$$

$$= \boxed{r^2 d\theta^2 + dr^2}$$

restrict to $S^1(\mathbb{R})$, so r is not changing,

$$\text{so we have } g_{S^1} = \underline{r^2 d\theta^2}, \quad G = [\mathbb{R}]$$

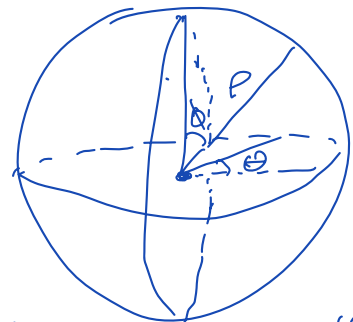
Since volume form is arclength form in 1D,
is $\int \det G \cdot \text{wedges} = d\text{vol}$

$$dS = d\text{vol} = \int r^2 d\theta^2 = \underline{r d\theta} \quad \left[\int_0^{2\pi} r d\theta = 2\pi r \right]$$

$g(d\theta, d\theta)$
if

Also if you use length $\theta = t, t \in (0, 2\pi]$ so $r'(t) = dr/dt = r$, $g(r', r') = r^2$
 $L = \int_0^{2\pi} \sqrt{g(r', r')} dt = \int_0^{2\pi} \sqrt{r^2} dt = 2\pi r$. they coincide on S^1 here but for
 S^2 volume form is surface area, length formula measures curve
 on sphere.

Sphere S^2 in $\mathbb{R}^3$.



ρ = distance from O , ϕ = angle from north pole,

θ = longitude

Express in spherical coordinates

$$\psi(\rho, \phi, \theta) = (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)$$

$$\psi^* (dx^2 + dy^2 + dz^2) = d\rho^2 + \rho^2 d\phi^2 + \rho^2 \sin^2 \phi d\theta^2$$

we restrict to $S^2(R)$ - $i: (\phi, \theta) \mapsto (R, \phi, \theta)$ with

$\rho = R$ fixed

$$i^* (d\rho^2) = 0, \quad i^* (\rho^2 d\phi^2) = R^2 d\phi^2$$

$$i^* (\rho^2 \sin^2 \phi d\theta^2) = R^2 \sin^2 \phi d\theta^2$$

$$= R^2 (d\phi^2 + \sin^2 \phi d\theta^2)$$

$$G = R^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \phi \end{pmatrix} \rightarrow \text{Riemannian metric}$$

volume form : $\sqrt{\det g} = d\phi \wedge d\theta$

fix θ : meridian
 $g = R^2 d\phi^2$, circle radius R .
 fix ϕ : latitude circle,
 $g = (R \sin\theta)^2 d\theta^2$,
 so circle radius $R \sin\theta$

$$= \int R^4 \sin^2\phi \, d\phi \wedge d\theta$$

$$= R^2 \sin\theta \, d\phi \wedge d\theta = d\text{vol}$$

$$\int_0^{2\pi} \int_0^\pi R^2 \sin\theta \, d\phi d\theta = 4\pi R^2 \rightarrow \text{surface area of sphere}$$

4. Final Closing thoughts

if we have inner product on vector space $\langle -, - \rangle$,
 then we get an isomorphism $V \xrightarrow{\sim} V^*$ because
 we can just fix $v \in V$, and as $v \mapsto \langle v, - \rangle$, which
 is a linear map on a vector to $\mathbb{R}$, which is by definition an element
 of dual space.

if we have Riemannian metric on every tangent space,
 then at each $p \in M$, we have iso $T_p(M) \rightarrow T_p^*(M)$

because $v_p \mapsto g_p(v_p, -)$ for $v_p \in T_p(M)$

So vector fields $\xrightarrow{\sim}$ 1-forms, $v \mapsto g(v, -)$ in general,
 so can go from forms to vector fields too.

(different metric means different isomorphism). Apdx 1.

$$f \rightarrow df \xrightarrow{\sim} \text{Grad } f \rightarrow \text{by definition}$$

eg. $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$, $\text{grad } f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$.

by definition $g(\text{Grad } f, v) = df(v) = v(f)$

here $df(v) = v(f)$ is rate of f changes with v .

gradient is vector that paired with v using metric reproduces that rate.

Appendix

1. why isomorphic

$$V \xrightarrow{\phi} V^*$$

$$\dim_n \quad \dim_n$$

$$v \mapsto \langle v, - \rangle$$

since we are linear

because $\langle -, - \rangle$ is linear in second slot, ϕ linear in first slot.

show $\ker \phi = \{0\}$, we will be an isomorphism

because we will have full rank, same dimension.

because inner product is positive definite we know with b is isomorphic. $v \in \ker \phi \Leftrightarrow \langle v, - \rangle = 0 \Leftrightarrow \langle v, w \rangle = 0 \forall w \Rightarrow \langle v, v \rangle = 0$ by pos def $v=0$

2. Spectral theorem: if $A^T = A$ then eigenvalues are real, and eigenvectors chosen to be orthogonal

$$\text{if } Av = \lambda v \quad \langle v, v \rangle = v^T (Av) = v^T (\lambda v) = \lambda (v^T v) = \lambda \|v\|^2 > 0$$

$$\lambda > 0$$

$$\text{Def. let } C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad w = c_1 \alpha_1 + c_2 \alpha_2$$

$$\text{by linearity } C^T G C = \sum_{i,j} c_i c_j \langle \alpha_i, \alpha_j \rangle = \langle w, w \rangle$$

the α 's are a basis so $C \neq 0$ means $w \neq 0$,

and by pos def $\langle w, w \rangle > 0$, G is pos def

by definition if $C^T G C > 0$ for every nonzero C .

