

Solutions to Quiz

October 21, 2026

1. Define smooth manifolds and smooth maps.

A topological space M is an n -dimensional manifold if every point $p \in M$ has an open neighborhood U with a homeomorphism $\phi : U \rightarrow \mathbf{R}^n$. It is a smooth manifold if for every pair $(U, \phi), (V, \psi)$ containing p , the map $\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V)$ is a smooth map between open subsets of $\mathbf{R}^n$.

A continuous map $F : M \rightarrow N$ between manifolds is called smooth if for any chart (U, ϕ) for a point $p \in M$ and (V, ψ) for $F(p) \in N$, the map $\psi \circ F \circ \phi^{-1} : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is smooth.

Note: there is a more flexible notion where we ask for ϕ to be a homeomorphism to any open subset of $\mathbf{R}^n$, but these are equivalent. Any open subset of $\mathbf{R}^n$ contains an open ball around $\phi(p)$. The chart can be restricted to the preimage of the ball and post-composed with an homeomorphism between the ball and $\mathbf{R}^n$.

2. Show that S^2 is a smooth manifold by constructing explicit charts.

Let S^2 be the points of $\mathbf{R}^3$ which are distance 1 from the origin with the usual metric. For $i = 1, 2, 3$, let U_i^+ denote the open subset $S^2 \cap \{x_i > 0\}$, and let U_i^- be $S^2 \cap \{x_i < 0\}$. The maps $U_i^\pm \rightarrow \mathbf{R}^2$ given by projecting onto the coordinates x_k, x_j which are distinct from x_i is a homeomorphism with inverse $(x_j, x_k, \pm \sqrt{1 - x_j^2 - x_k^2})$. The transition between $U_i^\pm$ and $U_j^\pm$ is $(x_k, \pm \sqrt{1 - x_j^2 - x_k^2})$, which is a smooth map.

3. State the level set theorem and use it to construct TS^2 .

If $F : M \rightarrow N$ is a smooth map with constant rank k in a neighborhood of $p \in N$, then the level set $F^{-1}(p) \subseteq M$ is a codimension k submanifold of M whose tangent bundle is the subbundle $DF^{-1}(p, 0)$ of TM .

Let $F : \mathbf{R}^3 \rightarrow \mathbf{R}$ be defined by $x^2 + y^2 + z^2$. The 2-sphere S^2 is $F^{-1}(1)$. The differential $DF : TR^3 \rightarrow TR$ is simply $[2x, 2y, 2z]$ which is rank 1 as long as $(x, y, z) \neq 0$. Thus S^2 is a submanifold of dimension $3 - 1 = 2$. The kernel of DF is precisely the set of vectors $v \in T_{(x,y,z)}\mathbf{R}^3$ such that $v \cdot (x, y, z) = 0$. In other words, the plane which is orthogonal to the vector connecting the origin to the point on S^2 .

4. Give an example of a non-trivial smooth vector field on the torus. Compute its integral curves and its flow.

Let $T = S^1 \times S^1$ be given coordinates θ, ϕ . The constant section ∂_θ defines a non-vanishing vector field on S^1 , so it also defines one on T . Its maximal integral curves are the speed 1 paths which follow $S^1 \times \{\phi\}$ for each $\phi \in S^1$ in the clockwise direction. Its flow is speed 1 clockwise rotation around the first S^1 factor.

5. Define and use the standard basis to describe all differential forms on $\mathbf{R}^4$.

The tangent bundle of $\mathbf{R}^4$ is trivial, so we freely identify all of the tangent spaces with $\mathbf{R}^4$. Let $dx_i : \mathbf{R}^4 \rightarrow \mathbf{R}$ be the projection onto the i th coordinate. Let $dx_i \wedge dx_j : \bigwedge^2 \mathbf{R}^4 \rightarrow \mathbf{R}$ be the determinant of the projection onto the i th and j th coordinates. Let $dx_i \wedge dx_j \wedge dx_k : \bigwedge^3 \mathbf{R}^4 \rightarrow \mathbf{R}$ be the determinant of the projection onto the i th, j th, and k th coordinates. Finally, let $dx \wedge dy \wedge dz \wedge dw : \bigwedge^4 \mathbf{R}^4 \rightarrow \mathbf{R}$ be the determinant.

The 0-forms on $\mathbf{R}^4$ are simply C^∞ functions. The 1-forms are $C^\infty(\mathbf{R}^4)$ -linear combinations of dx, dy, dz, dw . The 2-forms are $C^\infty(\mathbf{R}^4)$ -linear combinations of $dx \wedge dy, dx \wedge dz, dx \wedge dw, dy \wedge dz, dy \wedge dw, dz \wedge dw, dw$. The 3-forms are $C^\infty(\mathbf{R}^4)$ -linear combinations of $dx \wedge dy \wedge dz, dx \wedge dz \wedge dw, dx \wedge dy \wedge dw, dx \wedge dy \wedge dz$. The 4-forms on $\mathbf{R}^4$ are simply $C^\infty(\mathbf{R}^4)$ multiples of $dx \wedge dy \wedge dz \wedge dw$. All k -forms with $k > 4$ are necessarily 0.

6. Let $\omega = z^2 dx \wedge dy + w dw \wedge dz$ be a 2-form on $\mathbf{R}^4$. Define $s : \mathbf{R}^3 \rightarrow \mathbf{R}^4$ by

$$\mathbf{x} = (x_1, x_2, x_3) \mapsto \left(\frac{2x_1}{|\mathbf{x}|^2 + 1}, \frac{2x_2}{|\mathbf{x}|^2 + 1}, \frac{2x_3}{|\mathbf{x}|^2 + 1}, \frac{|\mathbf{x}|^2 - 1}{|\mathbf{x}|^2 + 1} \right).$$

Compute $d\omega$ and $s^*\omega$.

The exterior derivative is $d\omega = 2z dx \wedge dy \wedge dz$. To compute the pullback we first compute the Jacobian:

$$Ds_{(x_1, x_2, x_3)} = \frac{2}{(|x|^2 + 1)^2} \begin{bmatrix} |\mathbf{x}|^2 + 1 - 2x_1^2 & 2x_1x_2 & 2x_1x_3 \\ 2x_1x_2 & |\mathbf{x}|^2 + 1 - 2x_2^2 & 2x_2x_3 \\ 2x_1x_3 & 2x_2x_3 & |\mathbf{x}|^2 + 1 - 2x_3^2 \\ 4x_1 & 4x_2 & 4x_3 \end{bmatrix}$$

In terms of the components s_{ij} of Ds we have

$$\begin{aligned} s^*\omega &= [s_3^2(s_{11}s_{22} - s_{12}s_{21}) + s_4(s_{41}s_{32} - s_{42}s_{31})]dx_1 \wedge dx_2 \\ &\quad + [s_3^2(s_{12}s_{23} - s_{13}s_{23}) + s_4(s_{42}s_{33} - s_{43}s_{33})]dx_2 \wedge dx_3 \\ &\quad + [s_3^2(s_{13}s_{21} - s_{11}s_{23}) + s_4(s_{43}s_{31} - s_{41}s_{33})]dx_3 \wedge dx_1 \end{aligned}$$

7. State Stokes' theorem. Let η be the 3-form $z dx \wedge dy \wedge dz$ restricted from $\mathbf{R}^4$ to S^3 . Compute $\int_{S^3} \eta$. Does η vanish anywhere? If so, where?

Let M be an oriented m -manifold with boundary and let ρ be a $(k-1)$ -form on M . Stokes' theorem states $\int_M d\rho = \int_{\partial M} \rho$.

The form η is simply $\frac{1}{2}d\omega$ where ω is as in the previous question so $\int_{S^2} \eta = \frac{1}{2} \int_{\emptyset} \omega = 0$.

The form η vanishes at all points where $z = 0$ and at all points where the tangent space has no z , x or y component. On $\mathbf{R}^3$ it vanishes everywhere on the xy -plane. On S^3 it vanishes at the four points in the xy -plane: $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and the two points where the tangent space has no z -component: $(0, 0, \pm 1)$.

8. Define the gradient of a function on a Riemannian manifold. Compute the gradient of $\sin : S^1(R) \rightarrow \mathbf{R}$ where $S^1(R)$ is the circle of radius R with the standard metric.

Let M be a Riemannian manifold with metric g and $f : M \rightarrow \mathbf{R}$ be a smooth function. The gradient of f is the unique vector field $\text{grad} f$ on M such that $g(\text{grad} f, X) = df(X)$ for any other vector field X .

The standard metric on $S^1(R)$ is $R^2 d\theta^2$ and the differential of $\sin \theta$ is $\cos \theta d\theta$. Suppose $X = h(\theta) \partial_\theta$ is a vector field on $S^1(R)$. $g(\frac{1}{R^2} \cos \theta \partial_\theta, h(\theta) \partial_\theta) = \frac{R^2}{R^2} \cos \theta h(\theta) = \cos \theta d\theta(h(\theta) \partial_\theta)$. Therefore $\text{grad} f = \frac{1}{R^2} \cos \theta \partial_\theta$.