

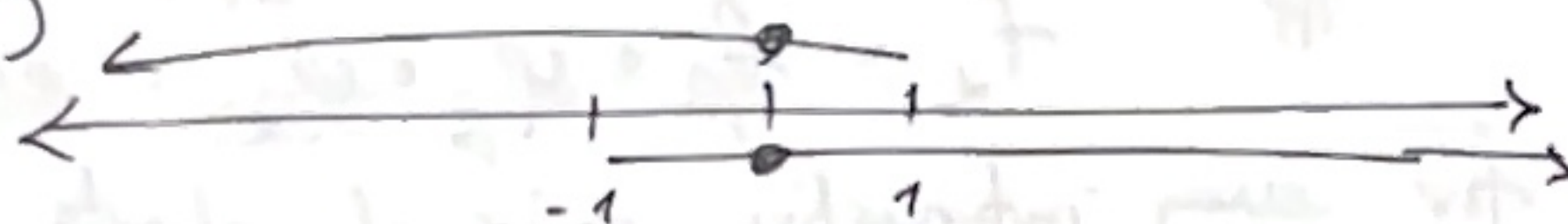
① Top & Physics Lecture 2: Smoothness & constructions. (Felix)

1. Smooth structures.

a. Defining func locally.

⊛ How do we define a function $\mathbb{R} \rightarrow \mathbb{R}$?

one way



first define it for everything ≤ 0

then

≥ 0

when does this give a func on $\mathbb{R}$?

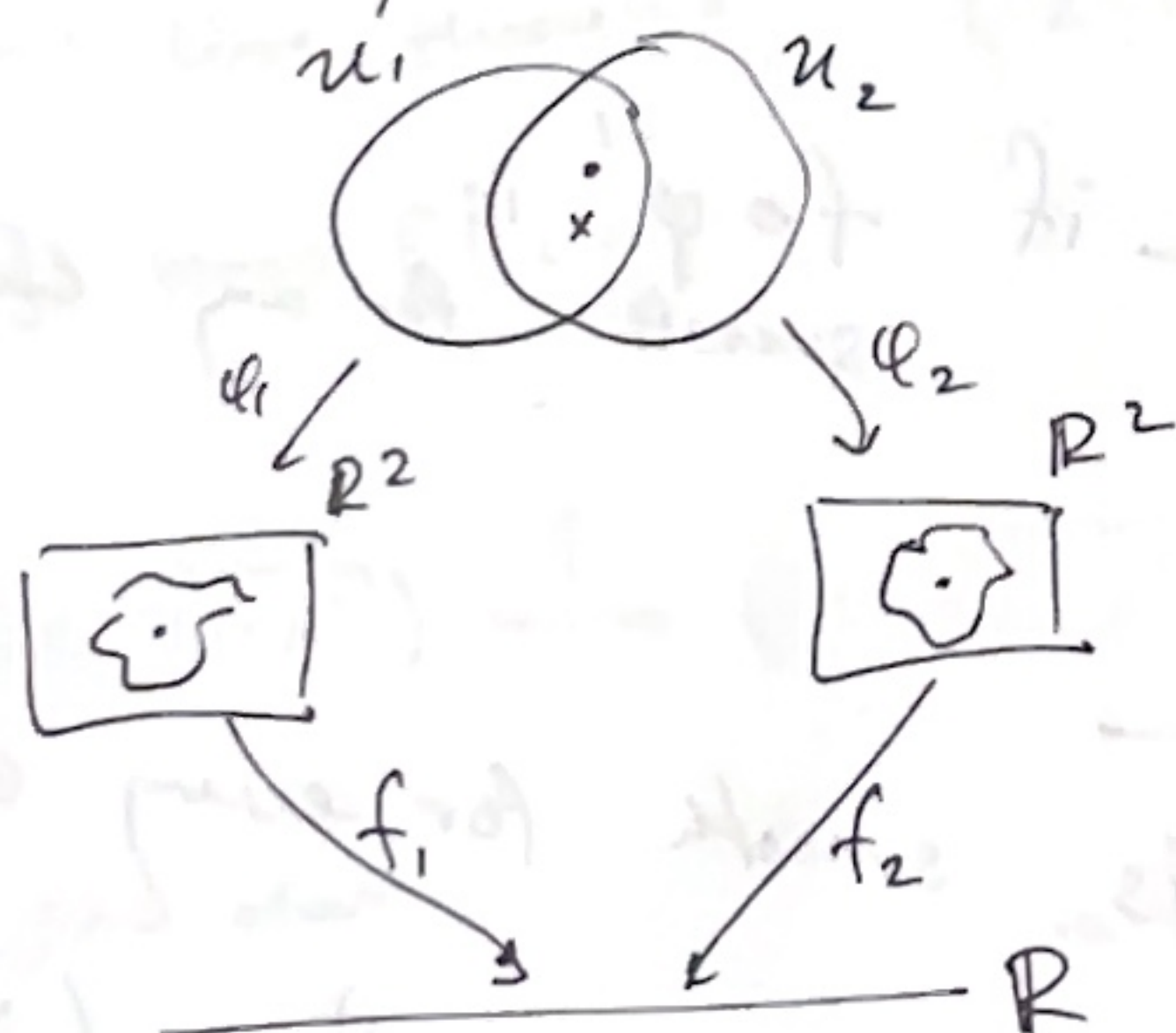
if $f_{\leq 0}(0) = f_{\geq 0}(0)$ (this is the most basic notion of a piecewise func or a clutching or gluing or patching etc.)

what if we have $f_{\leq 1}$ & $f_{\geq -1}$

the must agree on $[-1, 1]$

Now to manifolds

we have a manifold M which is covered by charts (U_x, φ_x)



can we define a func on $U_1 \cup U_2$ by defining Real func on $\varphi_1(U_1), \varphi_2(U_2) \subseteq \mathbb{R}^2$?

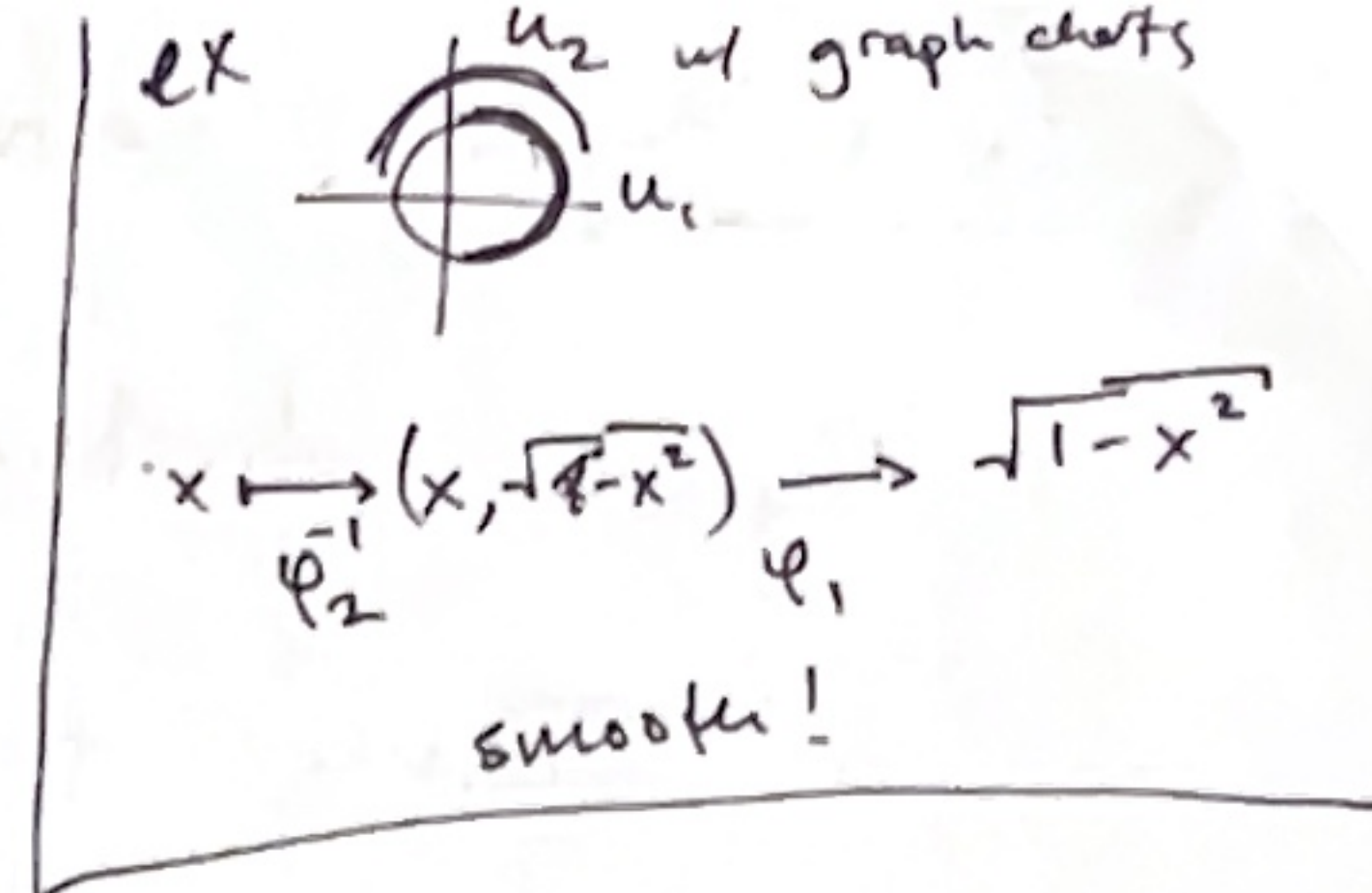
yes! we have a pair but they need to agree on the overlap

the only problem is $\varphi_1(U_1)$ & $\varphi_2(U_2)$ may not actually intersect. if $x \in U_1 \cap U_2$ what we need is that $f_1(\varphi_1(x)) = f_2(\varphi_2(x))$

we want our function to be defined only on subsets of $\mathbb{R}^2$ so the cond should be reformulated as: for $y \in \varphi_1(U_1 \cap U_2)$, $f_1(y) = f_2(\varphi_2 \circ \varphi_1^{-1}(x))$

(2)

$\varphi_2 \circ \varphi_1^{-1}$ is called the transition function
 ($\varphi_1 \circ \varphi_2^{-1}$ is its inverse)



So Prop given an atlas $(\mathcal{U}, \varphi_\alpha)$
 & funcs $f_\alpha: \mathcal{U}_\alpha \rightarrow \mathbb{R}$
 we can define a global func $f: M \rightarrow \mathbb{R}$
 iff $f_\alpha = f_\beta \circ \varphi_\beta \circ \varphi_\alpha^{-1}$ on $\varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$
 for every intersecting pair of charts.

b. What if our f_α are all smooth or $\varphi_\alpha(\mathcal{U}_\alpha) \subset \mathbb{R}^2$?
 Q. when should we call the glued func f smooth?

A. when $\varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth

Def if $\varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth, then $\mathcal{U}_\alpha + \mathcal{U}_\beta$ are
smoothly compatible

M is a smooth manifold if it has a cover by smoothly compatible charts (an Atlas)
 a smooth structure is a maximal atlas (i.e. all possible compat charts)

Def $f: M \rightarrow \mathbb{R}^n$
 is a smooth func if $f \circ \varphi^{-1}$ is smooth for any chart $(\mathcal{U}, \varphi)$

$f: M \rightarrow N$
 is a smooth map
 if $\psi \circ f \circ \varphi^{-1}$ is smooth for every chart $(\mathcal{U}, \varphi)$ on M
 & $(\mathcal{V}, \psi)$ on N

Lema ~~smooth map~~ constant maps, identity, inclusions, compositions etc are all smooth.
 of submanifolds

3)

c. examples.

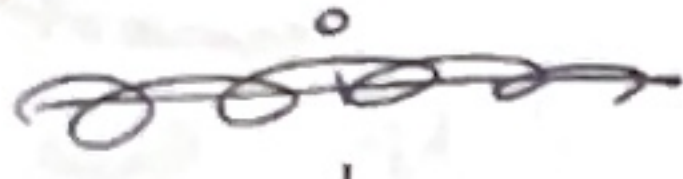
o) smooth maps between Euclidean space

$\begin{matrix} * \\ \text{pt} \end{matrix} \xrightarrow{f} M$

pick any chart for $f(x)$

locally this is inclusion of a point which is smooth.

1) $\mathbb{R} \xrightarrow{f} S^1 \subset \mathbb{R}^2$
 $t \mapsto (\cos t, \sin t)$



pick chart around 1

is the graph chart $(\{x > 0\}, \text{proj to } y\text{-axis})$

f is $t \mapsto \sin t$ smooth!

2) $S^1 \xrightarrow{i} \mathbb{R}^2$

near $(0,1)$ chart is $\varphi(x,y) = x$
 $\varphi^{-1}(x) = (x, \sqrt{1-x^2})$

smooth

3) $S^1 \xrightarrow{f} \mathbb{RP}^1$

$x \mapsto \text{line through } x$ (2:1 map)

$(x,y) \mapsto [x:y]$

$\begin{matrix} x \\ \mathbb{R}^1 \end{matrix} \xrightarrow{\varphi^{-1}} (x, \sqrt{1-x^2}) \xrightarrow{f} [x : \sqrt{1-x^2}] \xrightarrow{\phi} \frac{x}{\sqrt{1-x^2}} \in \mathbb{R}$

in $y > 0$ chart

in the $y \neq 0$ chart
(x might be zero here)

this is a smooth func!

4) New manifold $\mathbb{R} \cup \{\infty\} = \overline{\mathbb{R}}$

charts are $(\{x < \infty\}, \text{id})$ define $\frac{1}{\infty} := 0$ for this map.
 $\cup (\{x > 0\}, 1/x)$

④

$$\mathbb{RP}^1 \xrightarrow{f} \bar{\mathbb{R}}$$

$$[a:b] \mapsto \frac{b}{a} \quad (\text{define } \frac{a}{0} = \infty) \quad \text{well defined?}$$

in charts.

in $a \neq 0$ chart this is just the chart!

$$\mathbb{R}_0 \times \xrightarrow{\varphi_{a \neq 0}^{-1}} [1:x] \xrightarrow{f} \mathbb{R} \times \xrightarrow{\text{id}} \mathbb{R} \text{ smooth!}$$

in $b \neq 0$ chart.

$$x \xrightarrow{\varphi_{b \neq 0}^{-1}} [x:1] \xrightarrow{f} \frac{1}{x} \xrightarrow{1/x} x \text{ smooth!}$$

in the ∞ chart

in fact it has an inverse

$$\bar{\mathbb{R}} \longrightarrow \mathbb{RP}^1$$

$$x \mapsto [1:x]$$

$$\infty \mapsto [0:1]$$

why is this smooth?

in $x < \infty$ chart

$x > 0$

this computation inverse

so we have our first diffeomorphism!
(smooth w/ smooth inverse)

$$\bar{\mathbb{R}} \cong \mathbb{RP}^1$$

$$5) S^1 \xrightarrow{\quad} \bar{\mathbb{R}}$$

stereographic proj

$$\begin{array}{ccc} S^1 \setminus \{N\} & \xrightarrow{\quad} & \mathbb{R} \\ \downarrow & \nearrow & \downarrow \\ S^1 & \xrightarrow{N \mapsto \infty} & \mathbb{R} \end{array}$$

HW

check in charts that this is a diffeo

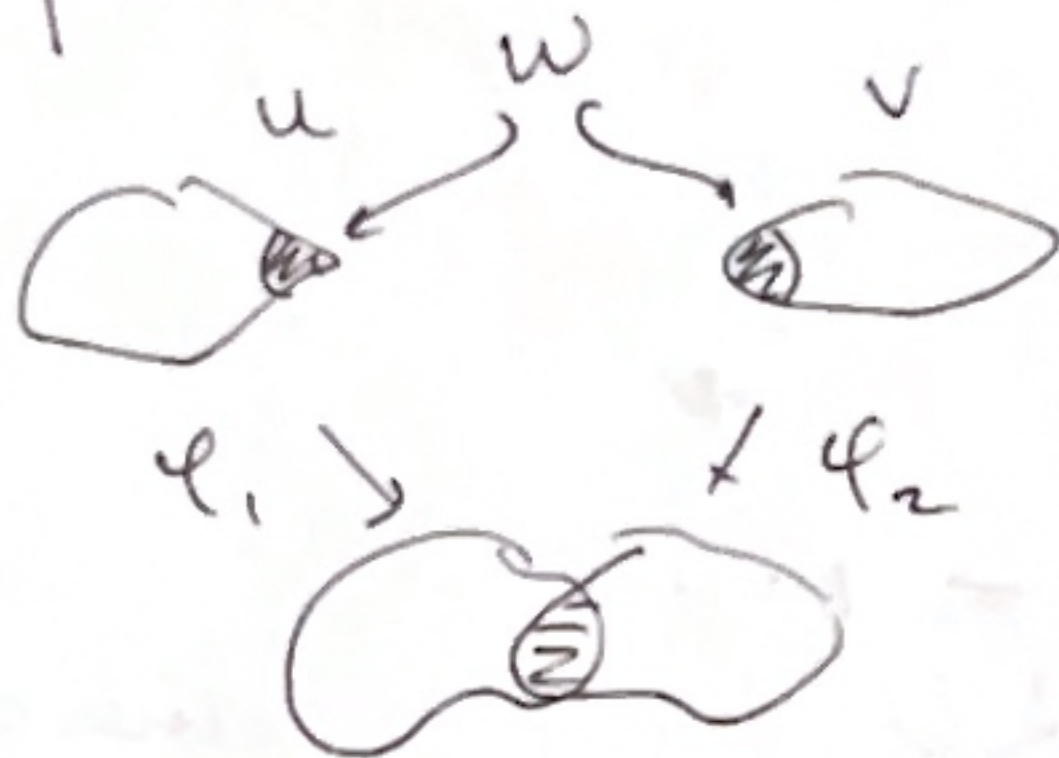
$$\therefore S^1 \cong \bar{\mathbb{R}} \cong \mathbb{RP}^1!$$

5

2. Gluing constructions.

Do the procedure in reverse:

start w/



"glue them" into a manifold along a diffeo morphism which will become a transition func.

Prop

$U, V \subset \mathbb{R}^n$ open

$W \subseteq U, W' \subseteq V$ open

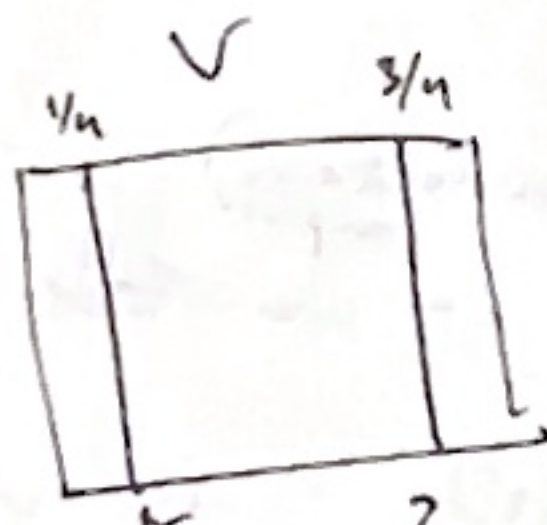
$\phi: W \rightarrow W'$ a diffeo morphism

then $U \sqcup V / \sim$ is a smooth manifold

w/ charts $(U, i_U^{-1}), (V, i_V^{-1})$ & transition func ϕ

Ex

Annulus / cylinder



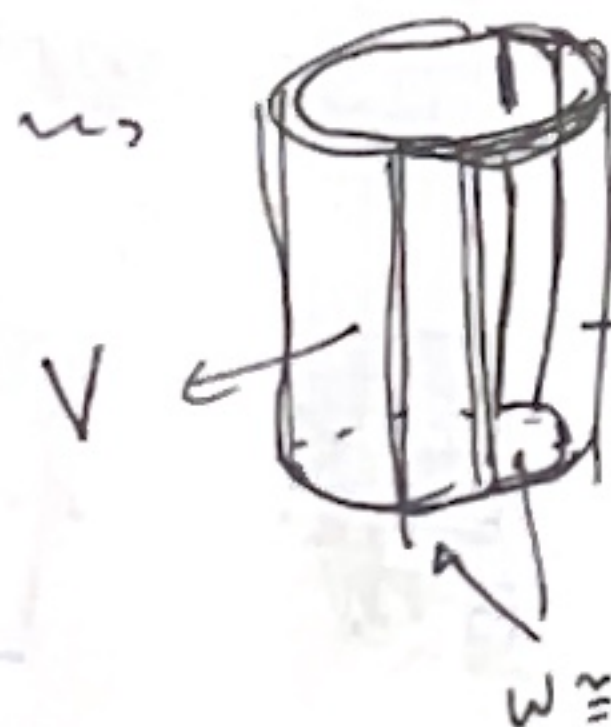
U, V both $[0, 1] \times [0, 1]$

$\phi: W \rightarrow W'$

$(t, x) \mapsto (1/4 - t, x)$ if $t \leq 1/4$

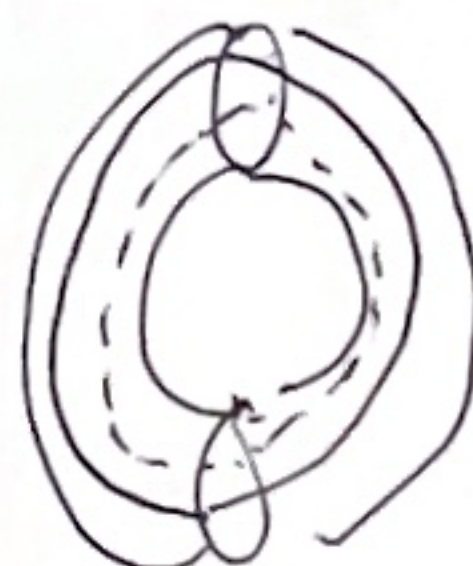
$\mapsto (1 - 3/4 - t, x)$ if $t > 3/4$

needs to be open



$= S^1 \times I$

Torus: also glue top to bottom two annuli together.

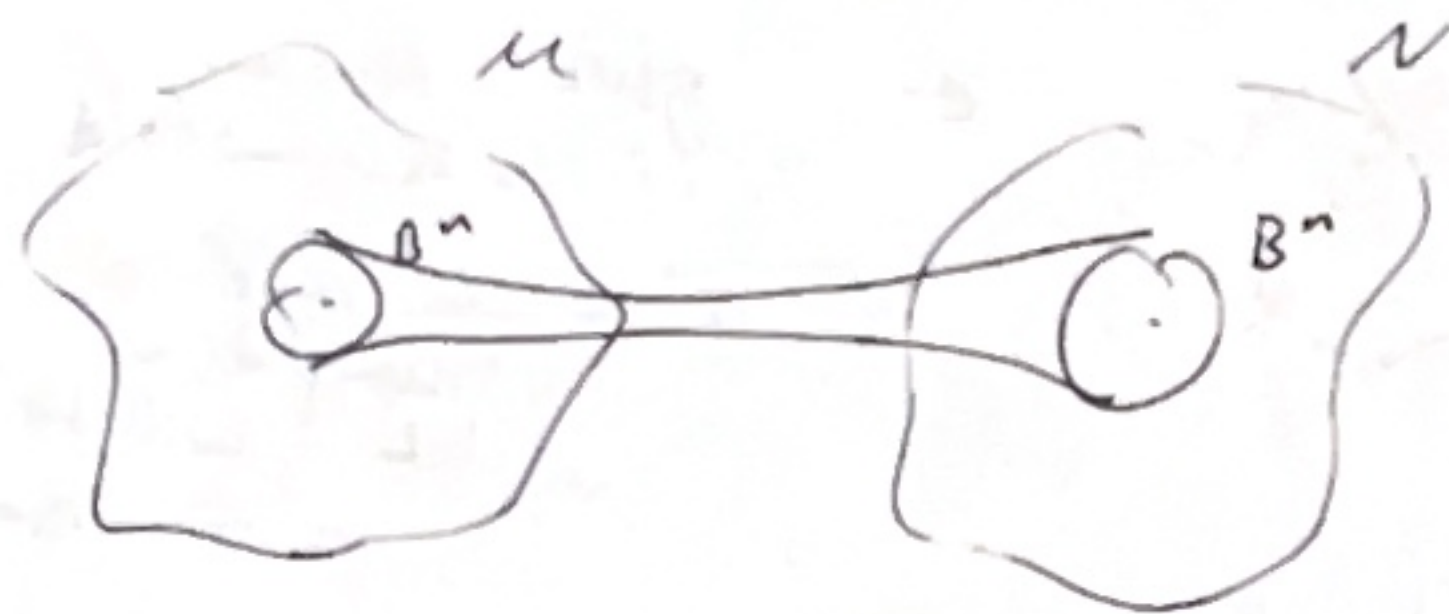


4 charts.

6) 3. ~~total~~

~~Products~~

Connect sums, $M^n + N^n$



remove B^n from each
+ glue together ∂B^n
along $S^{n-1} = \partial B^n$

Products

$M^n \times N^m$
charts $(U \times V, \varphi \times \phi)$

ex. $S^1 \times S^1$ the torus. HW prove that $S^1 \times S^1$ has the charts above.
+ $S^1 \times I$ (annulus)

The Classification of ^(connected compact) 0, 1, 2-manifolds.

0. every 0-manifold is diffeo (homeo) to a point $\mathbb{R}^0$
1. 1-man is $\cong S^1$
2. 2-man is $\cong S^2$
or $\cong \# S^1 \times S^1$
or $\cong \# \mathbb{R}P^2 = \mathbb{R}P^2 \# \mathbb{R}P^2$

can construct $\mathbb{R}P^2$ from an annulus

