Notes for Fall 2021 DRP:
Fundamental Groupoid and Van Kampen's Theorem

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1 Fundamental Groupoid

Algebraic Topology and more generally Homotopy Theory is about deformations of functions.

Starting with a very simple example, given any space X there is a natural correspondence between points in our space and maps from the one point space into it: $X \cong Hom(*, X)$.

This leads us to our first and simplest form of homotopy which is a continuous deformation of points:

A homotopy between maps $* \mapsto a$ and $* \mapsto b$ also known as a path between a and b is a continuous map $p : * \times I \cong I \to X$ such that $p(*, 0) = a$ and $p(*, 1) = b$.

Considering classes of points which are homotopic aka connected by a path, gives us our first homotopy group: $\pi_0(X)$ which is the set of path connected components of a space.

The space of all paths in a space carries a natural composition structure by traversing one path followed by another.

This gives us a second construction, the path graph $P(X)$ of a space (my notation). The path graph is the collection of all points in our space with arrows between them representing paths, and a composition operation connecting them.

However, this structure is missing many important properties. For example, the composition that we described above is not associative nor does it have proper inverses. Moreover, from the perspective of continuous deformation, there is a lot of extra information that we would like to reduce in a systematic way.

This gives rise to a more general notion of homotopy of paths that builds on the previous homotopy of points. If p and q are two paths from $a \to b$ in X, then a continuous map $h : I \times I \to X$ is a homotopy of paths if $h(x, 0) = p(x)$, $h(x, 1) = q(x)$, $h(0, t) = a$ and $h(1, t) = b$. We then use homotopy to reduce the complexity of our path graph into the fundamental groupoid $\Pi(X)$ of a space X. This structure consists of objects representing points, and arrows representing classes of homotopy equivalent paths between them. The composition product from before is now associative so we call it a category, and all arrows are invertible so we call it a groupoid. This construction has a natural action on continuous maps so we call it a functor from spaces to groupoids.

If we focus on a specific point and its automorphisms, we get a group which we call the fundamental group at that point $\pi_1(X, x)$. Note that if we choose the right topology (compact-open) to turn the set of loops into a space, these are just connected components.

2 Van Kampen's Theorem

In order to actually compute $\Pi(X)$, we often are unable to do it directly. The most basic tool is Van Kampen's Theorem which describes the fundamental groupoid of a space built out of an covering of path connected open subsets.

In its most basic form, an open covering is a collection of open sets whose union is the whole space. However, in order to translate this set theoretic idea into the world of groupoids, we need to describe open covers diagrammatically.

Given an open cover $\mathcal{U}$ we can make a diagram whose points are its open sets and whose arrows are inclusions between them. We call this a poset category and we can construct it out of any collection of open sets.

If we add all of the finite intersections of open sets to our diagram and then "glue" these open sets together we form the colimit of the diagram. In this case, the colimit is just the union of the covering.

More generally, the colimit object of the diagram uniquely maps into any other object that the diagram maps into. Of course, I don't really have time to explain all of these details precisely so just imagine "gluing" objects together in a way that preserves the relationships described by a diagram. Other examples of colimits: disjoint unions, direct sums, quotients, least upper bounds.

Van Kampen's Theorem tells us that the relationship described by a covering diagram is the same relationship that describes the fundamental groupoids associated with the open sets. More precisely: $\Pi(\text{colim}_{U \in \mathcal{U}} U) \cong \text{colim}_{U \in \mathcal{U}} \Pi(U)$. Intuitively this tells us that if we build a space by gluing together subspaces, we can build its fundamental groupoid by gluing as well. From this result, it follows "formally" that the same relationship holds between the fundamental groups.

3 Fundamental Group of the Circle

Now we'll show how this can be used in action to compute the fundamental group of S^1. If you look closely we're using covering spaces too.

To start we'll define a groupoid G that we will soon show is isomorphic to $\Pi(S^1)$. The objects of the groupoid are points in S^1 and the arrows are pairs (a, t) representing paths between a and $a \exp(2\pi i t)$. We'll abbreviate $\exp(2\pi i t)$ as $s(t)$.

G is a really nice algebraic description of $\Pi(S^1)$. In particular, the automorphisms of a point are indexed by integers and composition corresponds to addition, so $\pi_1(S^1, *) \cong \mathbf{Z}$

Next we consider the map $\zeta : G \to \Pi(S^1)$ which sends points to themselves and arrows (a, t) to the class of paths $x \mapsto as(tx)$ from $a \to as(t)$. The idea is that the arrows in G are nice representatives for the paths in S^1.

To show that ζ is an isomorphism we'll use Van Kampen with the covering $X_0 = S^1 \backslash \{1\} \subset \mathbf{C}$ and $X_1 = S^1 \backslash \{-1\} \subset \mathbf{C}$. Since these two spaces are in bijection with intervals on the line, they are contractible so their fundamental groupoids have one arrow between any pair of points. This means we can easily construct maps from each into G using the exponential function and apply Van Kampen's theorem to get a unique map $\gamma : \Pi(S^1) \to G$ that is a right inverse to ζ.

A bit of book keeping shows that γ is also a left inverse so that $\Pi(S^1) \cong G$ since any arrow in G can be broken up into pieces that lie in the images of γ, but the details are left as an exercise to someone who has more than ten minutes to present. It's also in tom Dieck's book.