

Recall: A tangent space  $T_p M$  is a vector space of all point derivations of  $C_p^\infty(M)$  for  $p \in M$

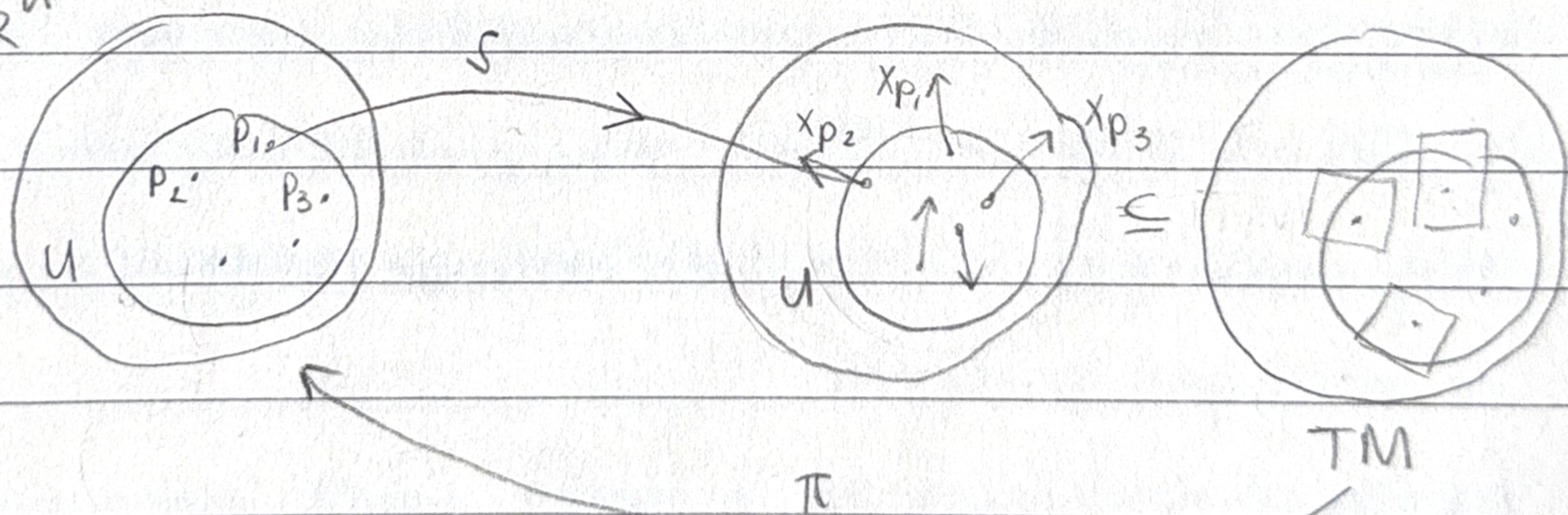
Def: A tangent bundle  $TM$  of  $M$  is the union of all tangent spaces of  $M$  (also denoted  $(TM, M, \pi)$  with  $\pi(v) = p$  if  $v \in T_p M$ )

$$TM = \bigcup_{p \in M} T_p M$$

Def: A section of a tangent bundle  $\pi: TM \rightarrow M$  is a map  $s: M \rightarrow TM$  st  $\pi \circ s = \text{id}_M$ , also called a vector field  $X$  of  $M$

Ex: Consider open  $U \subseteq \mathbb{R}^n$  with  $s$  assigning to each  $p \in U$  a tangent vector  $X_p \in T_p \mathbb{R}^n$

$$M = \mathbb{R}^n$$



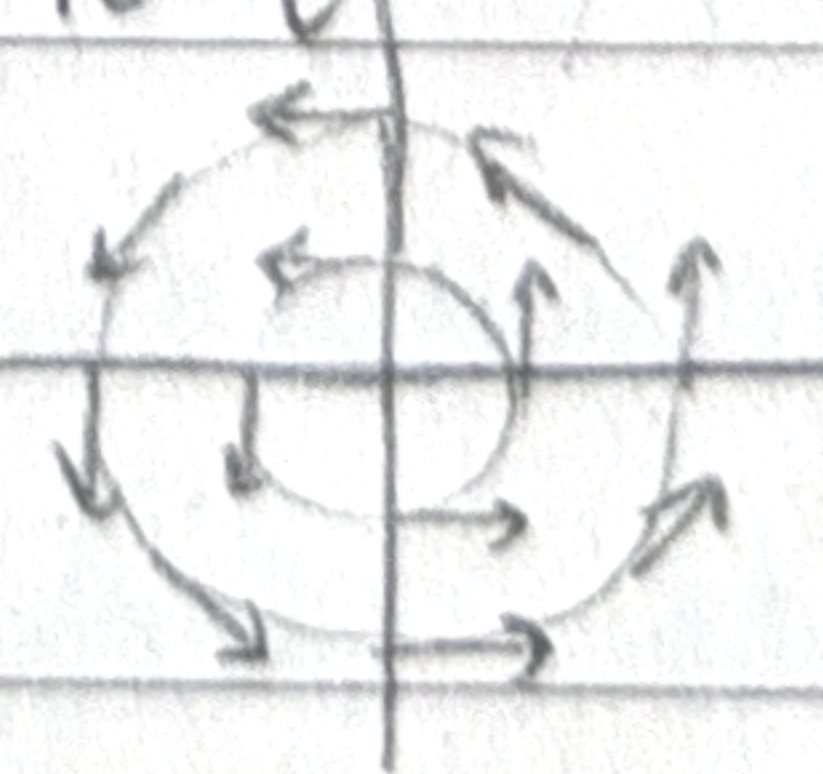
Here,  $X_p = \sum a^i(p) \frac{\partial}{\partial x^i} \Big|_p$ ,  $p \in U$ ,  $a^i(p) \in \mathbb{R}$

$$X = \sum a^i \frac{\partial}{\partial x^i}$$

with  $a^i$  representing the vector's component

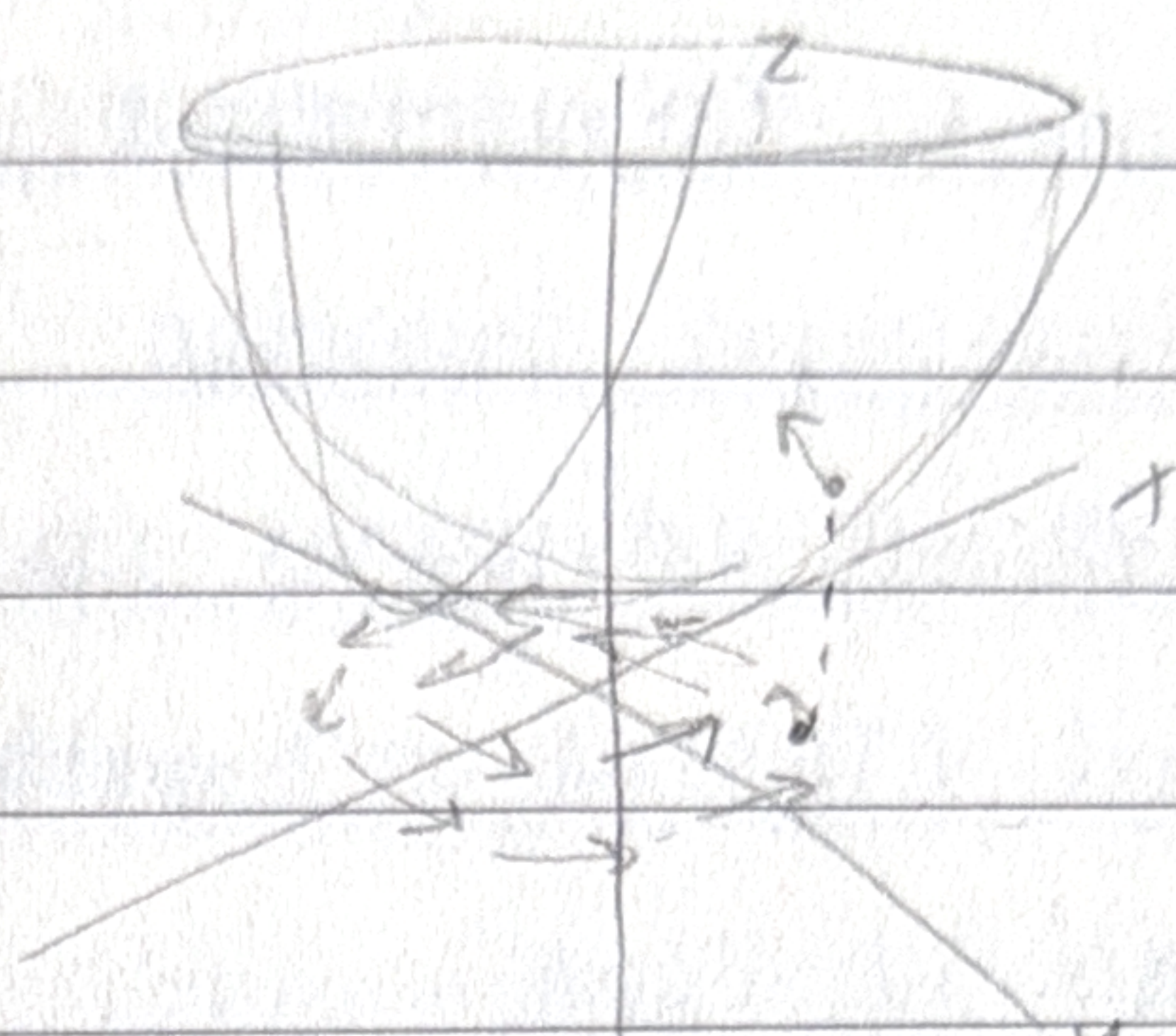
in direction  $i$ .  $X_p$  is  $C^\infty$  if  $a^i$  is  $C^\infty$

Ex:  $X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} = \begin{bmatrix} -y \\ x \end{bmatrix}$



We can even apply a function to get directional derivatives:  $(XF)(p) = \sum a^i(p) \frac{\partial F}{\partial x^i}(p)$

$$F = z = x^2 + y^2$$



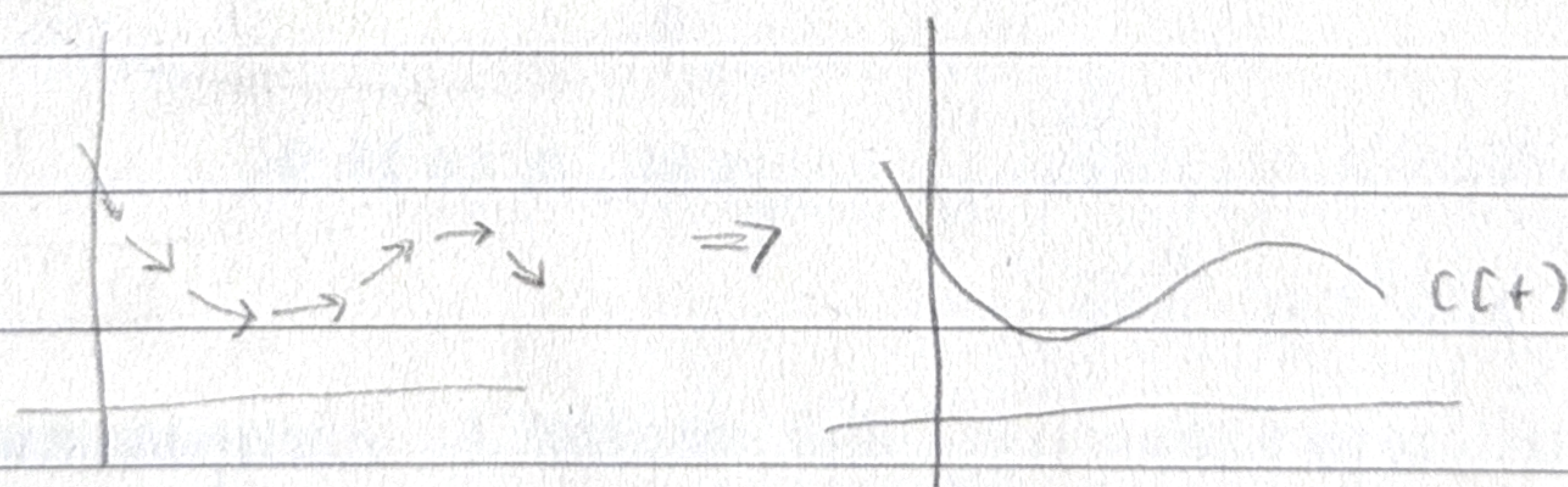
Note: When using coordinates  $X = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}$ , it's local to  $U$  since  $x^1, \dots, x^n$  only exist on a chart  $(U, \phi)$ ,  $U \subseteq M$ . On global manifolds, you rely on multiple charts sometimes

Def: Let  $X$  be  $C^\infty$  vector field on manifold  $M$ . An integral curve of  $X$  is a smooth curve

$$c: (a, b) \rightarrow M \text{ st } c'(t) = X_{c(t)} \quad \forall t \in (a, b).$$

It's maximal if  $(a, b)$  can't be extended to a larger interval

EX:



consider  $X$  smooth vector field on manifold  $M$  and choose starting point  $p$  and coordinate chart  $(U, \phi) =$

$(U, x^1, \dots, x^n)$  about  $p \in M$ . We can write

$X_{c(t)}$  in coordinate terms  $X_{c(t)} = \sum a^i(c(t)) \frac{\partial}{\partial x^i} \Big|_{c(t)}$

with  $c'(t) = \sum c'^i(t) \frac{\partial}{\partial x^i} \Big|_{c(t)}$ .

Applying the Picard-Lindelöf theorem that a first order ODE initial value problem

has a single unique solution on a local interval, we get:

Thm: Open  $V \subseteq \mathbb{R}^n$ ,  $p \in V$ ,  $f: V \rightarrow \mathbb{R}^n$  a  $C^\infty$  function. Then  $\frac{dy}{dt} = f(y)$ ,  $y(0) = p$  has a unique  $C^\infty$  solution  $y: (a, b) \rightarrow V$  where  $(a, b)$  is the maximal interval containing 0 on which  $y$  is defined.

Then we can define map on nbhd  $W$  of  $p$ ,

$F: (-\varepsilon, \varepsilon) \times V \rightarrow U$  for  $\varepsilon \in \mathbb{R}$  st  $\forall q \in V$ ,  $F(t, q)$

is an integral curve starting at  $q$ . This gives

us

Def: A local flow about  $p$  in open set  $V \subseteq M$  is

a  $C^\infty$  function  $F: (-\varepsilon, \varepsilon) \times W \rightarrow U$  st

i)  $F(0, q) = q \quad \forall q \in W$

ii)  $F_t(F_s(q)) = F_{t+s}(q)$  for  $t, s, t+s \in (-\varepsilon, \varepsilon)$

