AN ACTION OF THE WITT ALGEBRA ON KHOVANOV–ROZANSKY HOMOLOGY

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Abstract. We construct an action of the positive part of the Witt algebra on Khovanov–Rozansky $\mathfrak{gl}_N$-link homology and show that link cobordisms induce equivariant maps between twists of the homology. Moreover, the state spaces of simple webs are identified with standard representations of the Witt algebra on polynomials. Some simple relations to Lee homology and genus bounds are derived from the analysis of this presentation.

Contents

1. Introduction

Symmetries of link homology have been a consistently useful tool in the literature [GOR13, GHM24, CG24, EQ23, GLW18, KS25]. In this paper, we study differential operators on $\mathfrak{gl}_N$-homology following the work of Qi, Robert, Sussan, and Wagner [QRSW24]. Specifically, we consider an action by the Lie algebra $\mathfrak{W}_{-1}$, which we call the positive half of the Witt algebra, consisting of derivations on polynomials in one variable. In other contexts, $\mathfrak{W}_{-1}$ is referred to as a Cartan-type Lie algebra and denoted W_1 [Rud74]. Witt actions were introduced by Khovanov and Rozansky on triply graded homology via the standard action of $\mathfrak{W}_{-1}$ by derivations on Soergel bimodules [KR16]. A similar action was obtained in [QRSW24] by lifting the action on symmetric polynomials to foams via the Robert-Wagner evaluation formula. Using the foam action, it was shown that a copy of $\mathfrak{sl}_2$ embedded in $\mathfrak{W}_{-1}$ acts naturally on $\mathfrak{gl}_N$-homology [QRSW23]. In this paper we show that the full Lie algebra $\mathfrak{W}_{-1}$ acts naturally on $\mathfrak{gl}_N$-homology and in a forthcoming paper we extend the action to exterior power colored homology.

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Theorem 1.1. *Khovanov–Rozansky $\mathfrak{gl}_N$-homology is an isotopy-invariant representation of $\mathfrak{W}_{-1}$.*

Remark 1.2. In this paper Khovanov–Rozansky homology refers to the fully equivariant theory over the ring of symmetric polynomials, denoted R_N. To avoid conflicting terminology we will refer to this as R_N-linear homology and reserve the term *equivariant* for the $\mathfrak{W}_{-1}$-action.

The Lie algebra $\mathfrak{W}_{-1}$ is generated by a countable collection of operators $(L_k))_{k\geq -1}$ some of which have previously appeared in applications. Based on the results in [KR16], the action of the generator L_1 was used to define a p-DG structure on HOMFLY-PT homology and $\mathfrak{gl}_{-2}$-homology, leading to categorifications of Jones and colored Jones polynomial at a root of unity [QRSW21, QS22]. Similarly, the generator L_{-1} in [QRSW24] agrees with the operator ∇ introduced by Wang [Wan24] extending the work of Shumakovitch [Shu14].

The Witt action should be viewed as a generalization of the grading structure on ordinary Khovanov homology. The generator L_0 acts by multiplying homogeneous elements by their quantum degree. Therefore, the L_0-module structure is the grading and L_0-equivariance is grading-preservation. The familiar statement that the degree of a cobordism-induced map is its Euler characteristic has an analog for the $\mathfrak{W}_{-1}$ action. Grading shifts are replaced by local "twists" represented by red dots defined in Section 4.

Theorem 1.3. *Let $K_1^{\lambda_1}$ and $K_2^{\lambda_1}$ be knots decorated with red dots labelled λ_1 and λ_2 respectively. If $C : K_1^{\lambda_1} \to K_2^{\lambda_2}$ is a connected cobordism with $\lambda_2 - \lambda_1 = \chi(C)/2$, then*

$$\mathrm{KRW}_N(C) : \mathrm{KRW}_N(K_1^{\lambda_1}) \to \mathrm{KRW}_N(K_2^{\lambda_2})$$

is R_N-linear and $\mathfrak{W}_{-1}$-equivariant.

In simple cases the $\mathfrak{W}_{-1}$-structure of the homology can be identified with standard representations.

Proposition 1.4. *Let U be the unknot and let U^λ be the unknot with a red dot labeled λ. As $\mathrm{R}[x,y]^{S_2}$-modules and $\mathfrak{W}_{-1}$-representations,*

$$\mathrm{KRW}_2(\varnothing) \cong \mathrm{R}[x,y]^{S_2}, \tag{1}$$

$$\mathrm{KRW}_2(U) \cong (x-y)^{-1/2}\mathrm{R}[x,y], \tag{2}$$

$$\mathrm{KRW}_2(U^\lambda) \cong (x-y)^{(\lambda-1)/2}\mathrm{R}[x,y]. \tag{3}$$

In Section 7 we determine the structure of these modules. In particular, the only trivial submodule of $(x-y)^{(\lambda-1)/2}\mathrm{R}[x,y]$ is generated by 1 when it is present in the module. Therefore, the inclusion is the unique up-to-scalars map $\mathrm{KRW}_2(\varnothing) \to \mathrm{KRW}_2(U^\lambda)$ which is R_2-linear and $\mathfrak{W}_{-1}$-equivariant. Note that $(x-y)$ is the handle element and $(x-y)^{\pm 1/2}$ is the shift associated to a Reidemeister I move. These observations allude to the Rasmussen–Tanaka theorem and similar structural results in the literature [Ras05, Tan06, San20, Nao06, ILM25].

The structure of these representations are analyzed via an inner product on $\mathrm{R}[x,y]$, previously used in the work of Nishiyama [Nis96]. With respect to this inner product, the operator L_{-1} is adjoint to multiplication by $E_1 = x+y$. These considerations easily lead to the following proposition.

Proposition 1.5. *The kernel of L_{-1} on* KRW_2 *is naturally isomorphic to the $(X^2 - t)$-deformation of Khovanov homology, otherwise known as universal Lee theory. On the other hand, the $(X^2 - hX)$-deformation of Khovanov homology, otherwise known as Bar-Natan homology, is a quotient representation for $\mathfrak{W}_0$, the subalgebra generated by $(L_k)_{k\geq 0}$. The quotient has a splitting which carries a full $\mathfrak{W}_{-1}$-action, but the splitting is not natural with respect to cobordisms.*

The higher Witt operators also admit interesting adjoints but unfortunately the inner product does not have an obvious extension to the twisted state spaces.

Question 1.6. *Can the inner product and the adjoint operators defined in Section 7.2 be extend to twisted state spaces and link homology?*

These representations are independently interesting because the state spaces of unlinks and disjoint unions of theta webs are certain singular Bott-Samuelson bimodules. If grading shifts are replaced by twists of the same degree, then an interesting action by differential operators is obtained which is sensitive to grading. A distinct action of $\mathfrak{sl}_2$ on Soergel bimodules by multiplication operators was previously studied in the work of Elias and Williamson [EW14].

Question 1.7. *What is the relationship between the $\mathfrak{W}_{-1}$ operators and $\mathfrak{sl}_2$ operators on Soergel bimodules? Do the $\mathfrak{sl}_2$ operators of Elias and Williamson act on link homology?*

Finally, we remark that the $\mathfrak{W}_{-1}$ action has an interesting connection to geometry. The algebras of decorations for a generalized theta webs can be realized geometrically as the $U(N)$-equivariant cohomology of a flag variety in $\mathbb{C}^N$. The commuting action of S_N is the action of the Weyl group and the positive Witt operators are integral lifts of certain cohomology operations. More specifically, the rational algebra generated by $(L_k)_{k\geq 1}$ under composition is isomorphic to the Landweber-Novikov algebra of cohomology operations on complex cobordism and the mod p reduction of a special subalgebra is isomorphic to the subalgebra of the mod p Steenrod algebra generated by power operations [Woo97]. Moreover, the action of $\mathfrak{W}_{-1}$ on polynomials agrees with the action of the Landweber-Novikov algebra on the cohomology of the corresponding flag variety [Len98]. Actions of the Steenrod algebra have previously been studied on categorified quantum groups and Soergel bimodules [BC18, Kit13]. Unfortunately, the L_{-1} action does not have a clear interpretation in this context since it has negative degree and is therefore unstable.

Question 1.8. *Is there a geometric interpretation for the L_{-1} operator and the twisted representations?*

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1.2. **Outline.** Sections 2 and 3 review the necessary algebraic and topological machinery. In Section 2 we define a framework for twisted representations, which are used to construct equivariant chain complexes, and the relative homotopy category, which is the target of the homology functor. These constructions are defined for a general Lie algebra $\mathfrak{g}$, a Hopf algebra H, and an associative algebra A, but in the rest of the paper they will be specialized to the Witt algebra, its universal enveloping algebra, and the ring of symmetric polynomials, respectively. Section 2 also contains a brief discussion of the ring of symmetric polynomials and the Witt algebra. Section 3 reviews webs, foams, and the $\mathfrak{W}_{-1}$-action, based on [QRSW24, QRSW23].

The remaining sections contain new results with the main goal of defining the $\mathfrak{W}_{-1}$-equivariant Khovanov–Rozansky $\mathfrak{gl}_N$-homology. In Section 4, we describe twists of the $\mathfrak{W}_{-1}$-action on foams. Section 5 is devoted to defining the $\mathfrak{W}_{-1}$-module structure on Khovanov–Rozansky $\mathfrak{gl}_N$-homology and to proving that it is isotopy invariant. Section 6 discusses functoriality, which follows directly from results in the previous sections and from [ETW18]. In Section 7, we analyze the $\mathfrak{W}_{-1}$-structure of the state spaces of generalized theta webs and discuss how it behaves on specialization of parameters. We conclude in Section 8 by applying the results to the homology of $(2, k)$-torus links, extending the work of the second author [Roz26].

2. Algebraic Background

2.1. **Conventions.** Let N be a positive integer fixed throughout the paper. Let R be a unital ring in which 2 is invertible. For all $x \in \mathrm{R}$, define $\overline{x} := 1 - x$.

Let M_i denote the degree i subspace of a $\mathbb{Z}$-graded module M. Let $q^n M$ denote the shifted module where $(q^n M)_i = M_{i-n}$. Similarly if C is a chain complex, let $t^n C$ denote the shifted complex with $(t^n C)_i = C_{i-n}$. Complexes will be cohomologically graded. In other words, the ith differential goes from C_{i-1} to C_i.

2.2. **Symmetric Polynomials.** Let $\mathbf{a} = (a_1 + \cdots + a_k)$ be an integer composition of $n \in \mathbb{N}$, let $S_{\mathbf{a}} := S_{a_1} \times \cdots \times S_{a_k}$ be the associated Young subgroup of the symmetric group S_n, and let $\mathrm{R}_{\mathbf{a}} := \mathrm{R}[x_1, \cdots, x_N]^{S_{\mathbf{a}}}$ be its ring of invariants. In the special case where $a_1 = n$, $\mathrm{R}_{\mathbf{a}}$ is the ring of symmetric polynomials. When all $a_i = 1$, $\mathrm{R}_{\mathbf{a}}$ is the ring of all polynomials. The rings $\mathrm{R}_{\mathbf{a}}$ are graded by setting $\deg(x_i) = 2$.

In the ring R_n, let e_i, h_i, and p_i be the ith elementary, complete homogeneous, and power sum symmetric polynomials respectively. The elements of the ring R_N will be distinguished by capital letters. For example, the variables, the elementary, complete homogeneous, and power sum symmetric polynomials will be denoted X_i, E_i, H_i, and P_i respectively.

2.3. **The Witt algebra.**

Definition 2.1. The *Witt algebra* $\mathfrak{W}$ is the Lie algebra over R generated by $(L_n)_{n\in\mathbb{Z}}$ with the bracket

$$[L_n, L_m] := (n - m)L_{n+m}. \tag{4}$$

for all $n, m \in \mathbb{Z}$. For any integer $k \geq -1$, let $\mathfrak{W}_k$ denote the sub Lie algebra generated by $(L_n)_{n\geq k}$

Lemma 2.2. *The map $i : \mathfrak{sl}_2 \to \mathfrak{W}$ defined by*

$$i : \left| \begin{array}{ll} e & \mapsto L_{-1} \\ h & \mapsto 2L_0 \\ f & \mapsto -L_1 \end{array} \right. \tag{5}$$

is an injective map of Lie algebras.

Lemma 2.3. *The sub algebra* $\mathfrak{W}_{-1}$ *is the Lie algebra of derivations on* $\mathrm{R}[x]$ *under the standard representation*

$$L_n \mapsto -x^{n+1}\frac{\partial}{\partial x}. \tag{6}$$

The operators

$$L_n := -\sum_{i=1}^{k} x_i^{n+1}\frac{\partial}{\partial x_i} \tag{7}$$

define the tensor product action of $\mathfrak{W}_{-1}$ *on* $\mathrm{R}[x_1,\dots,x_k]$ *which commutes with the permutation action of* S_n*. In particular,* $\mathfrak{W}_{-1}$ *acts on the submodules* $\mathrm{R}_{\mathbf{a}}$*.*

2.4. **Twisted actions.** Let $\mathfrak{g}$ be a Lie algebra over R.

Definition 2.4. A $\mathfrak{g}$-*category*, is an R-linear category $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(V,W)$ is a $\mathfrak{g}$-representation and

$$X(f\circ g) = (Xf)\circ g + f\circ (Xg), \tag{8}$$

for all $X\in\mathfrak{g}$, $f\in \mathrm{Hom}_{\mathcal{C}}(W,U)$ and $g\in\mathrm{Hom}_{\mathcal{C}}(V,W)$.

Definition 2.5. A *Maurer–Cartan element* for an object V in a $\mathfrak{g}$-category $\mathcal{C}$ is an R-linear map $\alpha_{(-)}:\mathfrak{g}\to\mathrm{End}_{\mathcal{C}}(V)$ such that, for all $X,Y\in\mathfrak{g}$:

$$\alpha_{[X,Y]} = [\alpha_X,\alpha_Y] + X\alpha_Y - Y\alpha_X,$$

where $[-,-]$ in $\mathrm{End}_{\mathcal{C}}(V)$ is the commutator.

Lemma 2.6. *Let* (V,α) *and* (W,β) *be pairs of objects in* $\mathcal{C}$ *and Maurer–Cartan elements. The following defines a new* $\mathfrak{g}$*-action on* $\mathrm{Hom}_{\mathcal{C}}(V,W)$*:*

$$X\cdot f := Xf + \beta_X\circ f - f\circ\alpha_X \tag{9}$$

where Xf *denotes the original action of* X *on* $\mathrm{Hom}_{\mathcal{C}}(V,W)$*.*

Definition 2.7. The $\mathfrak{g}$-category $\mathrm{Tw}(\mathcal{C})$ is called the category of *twisted* $\mathcal{C}$*-objects*. Its objects are pairs of $\mathcal{C}$-objects and Maurer–Cartan elements. As R-modules

$$\mathrm{Hom}_{\mathrm{Tw}(\mathcal{C})}((V,\alpha),(W,\beta)) := \mathrm{Hom}_{\mathcal{C}}(V,W),$$

and the $\mathfrak{g}$-action is defined by equation (9).

2.5. **The relative homotopy category.** Let H be a Hopf algebra, let A an H-module algebra, and let $A\#H$ be their smash product. In the following, objects and morphisms in $A\#H$-mod will be called *H-equivariant* A-modules and morphisms. There is the forgetful functor between the homotopy categories of modules over these associative algebras

$$\mathrm{For}:\mathcal{C}(A\#H)\to\mathcal{C}(A).$$

An object in Ker(For) is called *relatively null-homotopic* and is characterized by being null-homotopic after forgetting the H structure.

Definition 2.8. The *relative homotopy category* is the Verdier quotient

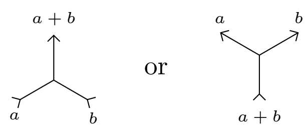
$$\mathcal{C}^H(A) := \frac{\mathcal{C}(A\#H)}{\mathrm{Ker}(\mathrm{For})}.$$

Lemma 2.9 ([QRSW21], Lemma 2.3)**.** *Let $M \xrightarrow{f} N \xrightarrow{g} L$ be a short exact sequence in $\mathcal{C}(A\#H)$ that is termwise A-split exact. If L is contractible, then f induces an isomorphism in $\mathcal{C}^H(A)$. Similarly, if M is contractible then g induces an isomorphism in $\mathcal{C}^H(A)$.*

3. The $\mathfrak{W}_{-1}$-action on Foams

3.1. **Webs and foams.** In this section, we review ($\mathfrak{gl}_N$-)foams, following [RW20] [QRSW23] and [QRSW24].

Definition 3.1. A *web* is a finite, oriented, trivalent graph $\Gamma = (V_\Gamma, E_\Gamma)$ embedded in $\mathbb{R}^2$ endowed with a *thickness function* $\ell : E(\Gamma) \to \mathbb{N}$. The embeddeding is required to be smooth away from vertices and the thickness function must satisfy a flow condition: every vertex must be one of two types

(10) 

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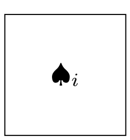
The first is called a *merge*, the second a *split*. In both cases, the edge with the largest label is called the *thick* edge and the other two are *thin* edges. Disjoint oriented circles without vertices are allowed.

Definition 3.2. A *foam* is a compact, finite, 2-dimensional CW complex F embedded in $\mathbb{R}^2 \times [0,1]$ together with a *thickness function* $\ell : F^2 \to \mathbb{N}$ and a *decoration*, $P_f \in \mathrm{R}_{\ell(f), N-\ell(f)}$, for every $f \in F^2$. The embedding is required to be smooth on the interiors of the 1 and 2-cells and every point p of F is required to have a closed neighbourhood homeomorphic to one of the following, shown in Figure 1:

(1) a disk, when p belongs to a unique facet,
(2) $Y \times [0,1]$, where Y is the neighbourhood of a merge or split vertex of a web, when p belongs to three facets,
(3) the cone over the 1-skeleton of a tetrahedron with p as the vertex of the cone, so that it belongs to six facets.

The 2-cells are called *facets*, the 1-cells are called *bindings*, and the 0-cells are called *singular vertices*. At a binding, the thicknesses of the three facets must satisfy a flow condition analogous to the one for webs. Orientations for bindings are induced by those of the thin facets and by the opposite of the thick facet. The *boundary* ∂F of F is the closure of the set of boundary points of facets that do not belong to a binding.

Let p_i denote the power sum in the first a variables in $\mathrm{R}_{a,N-a}$ and $\widehat{p}_i$ denote the power sum in the last $N-a$ variables in $\mathrm{R}_{a,N-a}$. The following foams are abbreviations for the these common decorations on a facet of thickness a:



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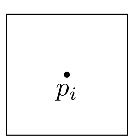
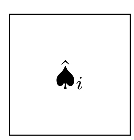


and


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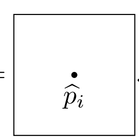

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In particular, $\spadesuit_0 = a$ and $\hat{\spadesuit}_0 = N - a$.

Partition F^1 as $F^1_\circ \sqcup F^1_-$, where $F^1_\circ$ is the collection of bindings diffeomorphic to circles, and F^1_- the ones diffeomorphic to intervals.

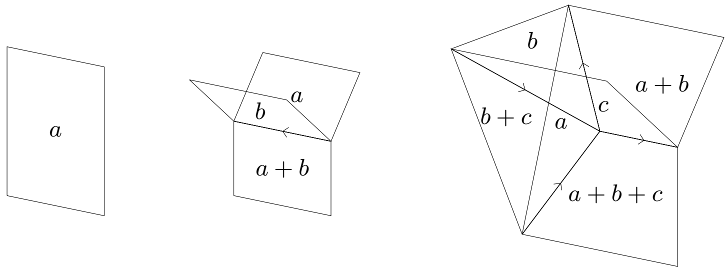


FIGURE 1. The three local models of the definition of foams. One denotes by $Y^{(a,b)}$ the one in the middle and $T^{(a,b,c)}$ the one on the right.

Definition 3.3. Let f be a facet of thickness a, let $b \in F^1_-$ have a neighbourhood homeomorphic to $Y^{(a,b)}$, and let $v \in F^0$ have a neighbourhood homeomorphic to $T^{(a,b,c)}$. The *degrees* of facets, bindings, singular vertices and foams are defined by

$$\deg_N(f) := a(N-a)\chi(f), \tag{11}$$

$$\deg_N(b) := ab + (a+b)(N-a-b), \tag{12}$$

$$\deg_N(v) := ab + bc + ac + (a+b+c)(N-a-b-c), \tag{13}$$

$$\deg_N(F) := \sum_{f \in F^2} (\deg(P_f) - \deg_N(f)) + \sum_{b \in F^1_-} \deg_N(s) - \sum_{v \in F^0} \deg_N(v) \tag{14}$$

Definition 3.4. A foam is called *basic* if it is equal to $\Gamma \times [0,1]$ outside a cylinder $B \times [0,1]$, and homeomorphic to one of the local models shown in Figure 2 within the cylinder. A foam is said to be in *good position* if it is a composition of basic foams and traces of isotopies.

Lemma 3.5 ([QW24])**.** *Every foam is isotopic to a foam in good position.*

A generic section $F_t := F \cap (\mathbb{R}^2 \times \{t\})$ is a web. In particular, the boundary of a foam F is a closed web in $\mathbb{R}^2$. Thus, one can define the following category.

Definition 3.6. The category $\overline{\mathbf{Foam}}$ has webs as objects and morphism sets

$$\overline{\mathbf{Foam}}(\Gamma_0, \Gamma_1) := {}^{\{\text{foams with } F_0 = -\Gamma_0 \text{ and } F_1 = \Gamma_1\}}\big/_{\text{ambient isotopy}},$$

where $-\Gamma_0$ denotes reversed orientations. Composition is given by stacking foams. The identity for Γ is the cylinder $\Gamma \times [0,1]$. Decorations are multiplicative and degrees are additive under composition. Let **Foam** be the graded, additive, R_N-linear completion of $\overline{\mathbf{Foam}}$.

Definition 3.7. Let $\langle - \rangle_N$ denote the R_N-valued Robert-Wagner *foam evaluation* of closed foams [RW20]. For any web Γ, define an R_N-bilinear form $\langle \cdot ; \cdot \rangle_N$ on $\mathbf{Foam}(\varnothing, \Gamma)$ by

$$\langle F; G \rangle_N := \langle \overline{G} \circ F \rangle_N, \tag{15}$$

where $\overline{G} : \Gamma \to \varnothing$ is the reflection of a foam $G : \varnothing \to \Gamma$ along the plane $\mathbb{R}^2 \times \{1/2\}$. The R_N-module

$$\mathcal{F}_N(\Gamma) := \mathbf{Foam}(\varnothing, \Gamma)/\mathrm{Rad}\langle \cdot ; \cdot \rangle_N \tag{16}$$

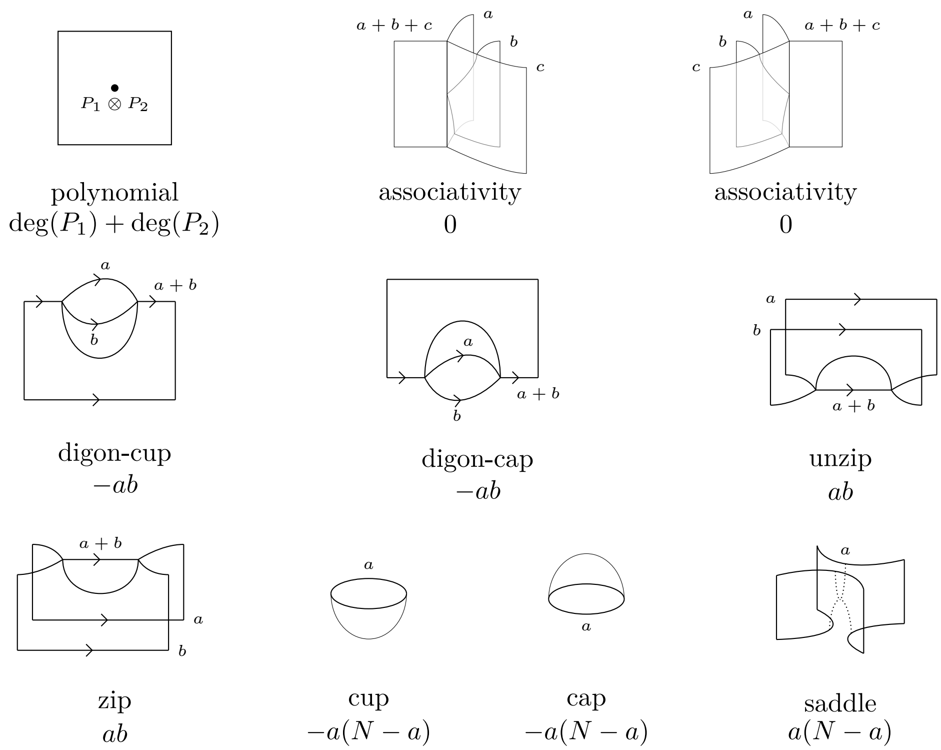
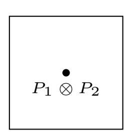
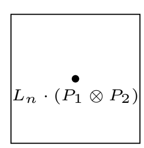
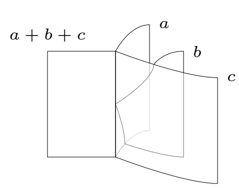
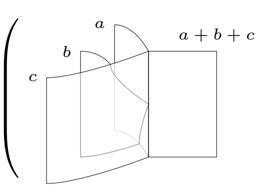


FIGURE 2. The degree of a basic foam is given below the name of each of the local models.

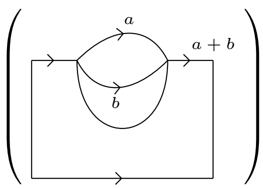
is called the *state space* of Γ. By the universal construction of [BHMV92] the state space extends to a functor $\mathcal{F}_N : \mathbf{Foam} \to \mathrm{R}_N$-mod.

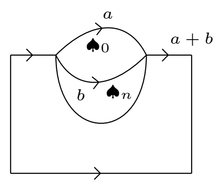
3.2. **The $\mathfrak{W}_{-1}$-action.** Fix three elements $s, \lambda, \mu \in \mathrm{R}$ and let $\lambda_n := \lambda(n+1)$ and $\mu_n := \mu(n+1)$[1]. Using these distinguished elements define the following operators $\mathbf{L}_n$ on basic foams:

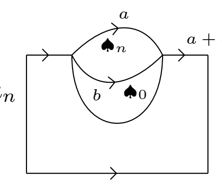
(17) $\mathbf{L}_n \Bigg($  $\Bigg) =$  ;

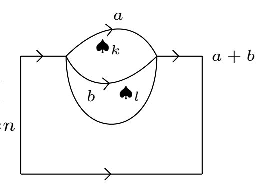
(18) $\mathbf{L}_n \Bigg($  $\Bigg) = \mathbf{L}_n \Bigg($  $\Bigg) = 0$;

[1]In [QRSW24], Witt sequences were used instead. When 2 is invertible in the ground ring, these sequences are of the form $(\lambda(n+1))_{n \in \mathbb{N}_{-1}}$ with $\lambda \in \mathrm{R}$.

(19) $\mathbf{L}_n$ 

$= \lambda_n$ 

$+ \mu_n$ 

$+ s \sum_{k+\ell=n}$ 

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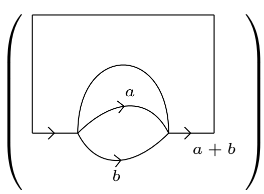
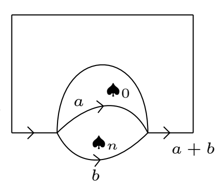
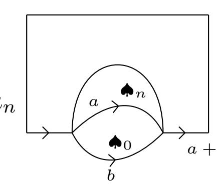
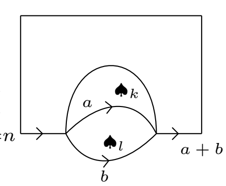
(20) $\mathbf{L}_n$ 

$= -\lambda_n$ 

$-\mu_n$ 

$+ \bar{s} \sum_{k+\ell=n}$ 

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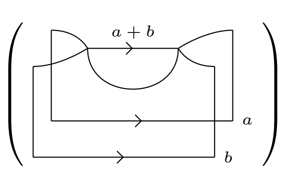
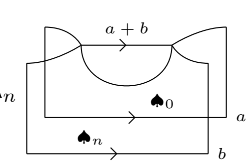
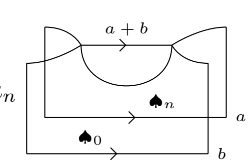
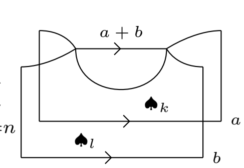
(21) $\mathbf{L}_n$ 

$= \lambda_n$ 

$+ \mu_n$ 

$- \bar{s} \sum_{k+\ell=n}$ 

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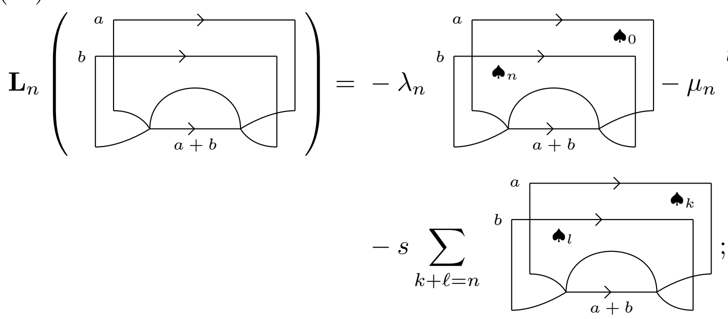
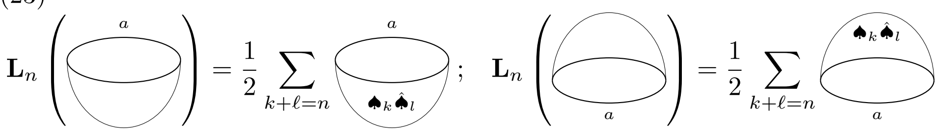
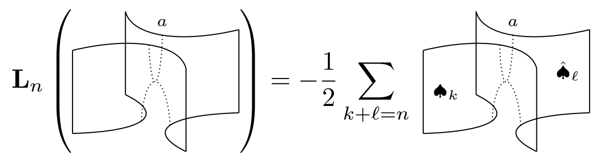
(22)



(23)



(24)



These operators are extended to arbitrary foams in good position by the Leibniz rule with respect to gluing and acting by 0 on traces of isotopies.

Theorem 3.8 ([QRSW24])**.** *The operators $\mathbf{L}_n$ define an action of $\mathfrak{W}_{-1}$ on the R_N-module generated by isotopy classes of foams in good position such that the Robert–Wagner evaluation on closed foams is an equivariant map.*

4. Twists of the $\mathfrak{W}_{-1}$-action

4.1. **Twists on webs.** In the language of Section 2.4, Theorem 3.8 implies that **Foam** is a $\mathfrak{W}_{-1}$-category.

Lemma 4.1. *A foam $F \in \mathbf{Foam}(\Gamma, \Gamma')$, induces a $\mathfrak{W}_{-1}$-equivariant map $\mathcal{F}_N(F) : \mathcal{F}_N(\Gamma) \to \mathcal{F}_N(\Gamma')$ if and only if $\mathfrak{W}_{-1}$ acts trivially on F.*

Proof. For any $X \in \mathfrak{W}_{-1}$ and $v \in \mathcal{F}_N(\Gamma)$, $F(X \cdot v) = X \cdot (Fv) - (X \cdot F)v$. Thus, a map $\mathcal{F}_N(F)$ intertwines the actions of $\mathfrak{W}_{-1}$ if and only if for all $X \in \mathfrak{W}_{-1}$, $X \cdot F = 0$. □

Corollary 4.2. *For any $F \in \mathrm{Hom}_{Tw(\mathbf{Foam})}((\Gamma, \tau), (\Gamma', \tau'))$, the induced map $\mathcal{F}_N(F) : \mathcal{F}_N(\Gamma, \tau) \to \mathcal{F}_N(\Gamma', \tau')$ is $\mathfrak{W}_{-1}$-equivariant if and only if, for every $L_n \in \mathfrak{W}_{-1}$*

$$\mathbf{L}_n F + \tau_{L_n} \circ F - F \circ \tau'_{L_n} = 0. \tag{25}$$

Definition 4.3. For each edge e in Γ with thickness a, let $D_e := \mathrm{R}_{a,N-a}$ be the algebra of polynomial decorations on that edge and let

$$D_\Gamma := \bigotimes_{e \in E(\Gamma)} D_e$$

be the algebra of all decorations on the web. Let $d : D_\Gamma \to \mathbf{Foam}(\Gamma, \Gamma)$. be the $\mathfrak{W}_{-1}$-equivariant map defined by sending a decoration to the cylinder, $\Gamma \times [0, 1]$, decorated by it.

Since the algebra of decorations is commutative, the condition for a Maurer–Cartan element α which lands in $d(D_\Gamma) \subset \mathbf{Foam}(\Gamma, \Gamma)$ reduces to

$$(n - m)\alpha_{L_{n+m}} = L_n \alpha_{L_m} - L_m \alpha_{L_n}.$$

In [QRSW23], this condition was called $\mathfrak{W}_{-1}$-*flatness* for the map

$$\alpha : \left| \begin{array}{ccc} \mathfrak{W}_{-1} & \longrightarrow & D_\Gamma \\ L_n & \longmapsto & \alpha_{L_n} \end{array} \right. .$$

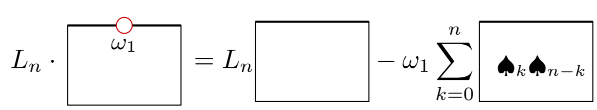
4.2. **Red dots.** The twist maps above are local in the sense that they involve decorations on a single edge. Therefore they can be conveniently tracked by a diagrammatic formalism. Let $p_k^{(e)}$ and $\widehat{p_k}^{(e)}$ denote the decorations p_k and $\widehat{p_k}$ in $R_{a,N-a} = D_e \subset D_\Gamma$.

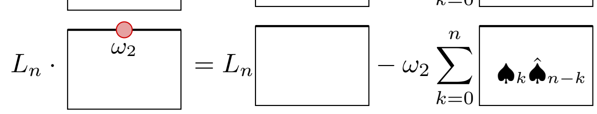
Definition 4.4. A *red-dotted* web is a web Γ endowed with a finite collection $\mathscr{D}$ of *red dots* of the following three types on the interiors of the edges of Γ:

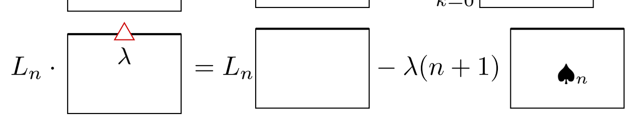
- A hollow red dot ○, associated to the twist
$$\tau_1 : L_n \mapsto \sum_{k=0}^{n} p_k^{(e)} p_{n-k}^{(e)}.$$
- A solid red dot ●, associated to the twist
$$\tau_2 : L_n \mapsto \sum_{k=0}^{n} p_k^{(e)} \widehat{p}_{n-k}^{(e)}.$$
- A triangular red dot △, associated to the twist
$$\tau_3 : L_n \mapsto (n + 1)p_n^{(e)}.$$

The linearity of twists implies that multiple red dots can lie on the same edge with R-valued labels representing linear combinations.

For three elements $\omega_1, \omega_2, \lambda \in$ R, the twisted action can be described by local pictures:

(26) 

(27) 

(28) 

Following Proposition 2.6, the twists on the target web are subtracted, while the twists on the source web are added.

Lemma 4.5. *The maps in Definition (4.4) define Maurer-Cartan elements.*

Proof. Since each twist lands in D_Γ, the Maurer-Cartan condition reduces to flatness. Assume without loss of generality that $n \leq m$. Then

$$\begin{aligned}
L_n \cdot \tau_1(L_m) - L_m \cdot \tau_1(L_n) &= \sum_{i=0}^{n} (m-n) p_i^{(e)} p_{n+m-i}^{(e)} + \sum_{i=n+1}^{m} (m-i) p_i^{(e)} p_{n+m-i}^{(e)} \\
&\quad - \sum_{i=n}^{m+n} (n-i) p_i^{(e)} p_{n+m-i}^{(e)} - \sum_{i=m}^{n+m} (i-m) p_i^{(e)} p_{n+m-i}^{(e)} \\
&= \sum_{i=0}^{n} (m-n) p_i^{(e)} p_{n+m-i}^{(e)} + \sum_{i=m}^{m+n} (m-n) p_i^{(e)} p_{n+m-i}^{(e)} \\
&\quad + \sum_{i=n}^{m-1} (i-n) p_i^{(e)} p_{n+m-i}^{(e)} + \sum_{i=n+1}^{m} (m-i) p_i^{(e)} p_{n+m-i}^{(e)} \\
&= (n-m)\tau_1(L_{m+n}).
\end{aligned}$$

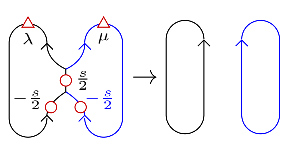
Thus τ_1 is flat. The computation for τ_2 is similar and is omitted. The computation showing that τ_3 is flat is as follows.

$$\begin{aligned}
L_n \cdot \tau_3(L_m) - L_m \cdot \tau_3(L_n) &= (m(m+1) - n(n+1)) p_{n+m}^{(e)} \\
&= (n-m)\tau_3(L_{m+n}).
\end{aligned}$$

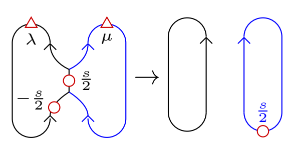
□

Remark 4.6. The triangular red dot labelled by $\frac{\lambda}{2} \in \mathrm{R}$ agrees with the hollow green dots labelled by λ for $\mathfrak{sl}_2$ defined in [QRSW23].

Example 4.7. The collection of twists which make a fixed foam equivariant is usually not unique. For example, the pair below of twists on the unzip foam induce $\mathfrak{W}_{-1}$-equivariant morphisms.



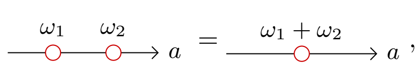
and



.

This illustrates that red dots can migrate between the source to the target along a shared facet while picking up a sign. The situation is analogous to the fact that grading shifts are relative and not absolute. Since twists are local, more variation is possible.

Lemma 4.8. *The Leibniz rule implies the following relations between twists:*

(29) 

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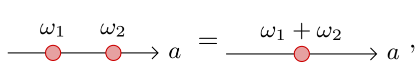
(30) 

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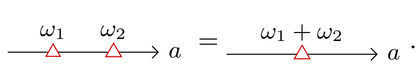
(31) 

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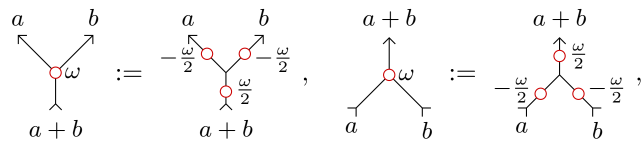
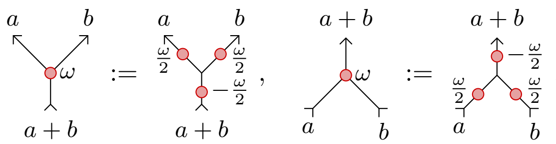
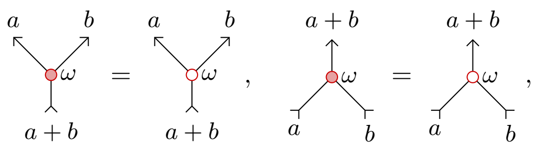
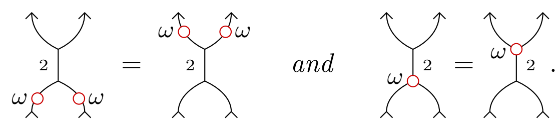
Definition 4.9. The following red dots abbreviate twists around trivalent vertices.

(32)


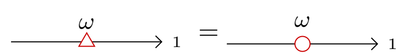
(33)


Lemma 4.10. *Dot migration rules* [RW20, (11)] *imply the following relation between twists:*

(34)


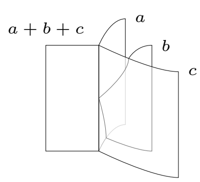
(35)


Lemma 4.11. *On a facet of thickness one, the dot twist reduces to the triangle twist:*

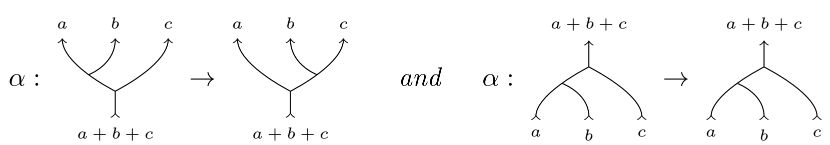
(36)


4.3. **Useful morphisms.**

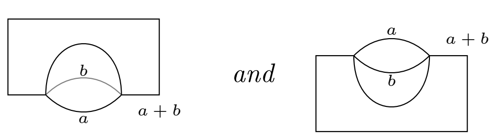
Lemma 4.12. *The two orientations for the following foam:*



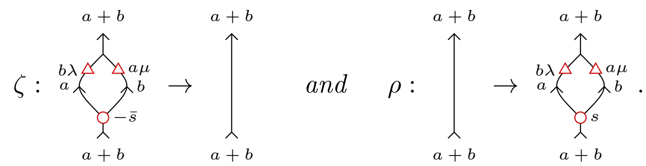
induce $\mathfrak{W}_{-1}$*-equivariant isomorphisms:*



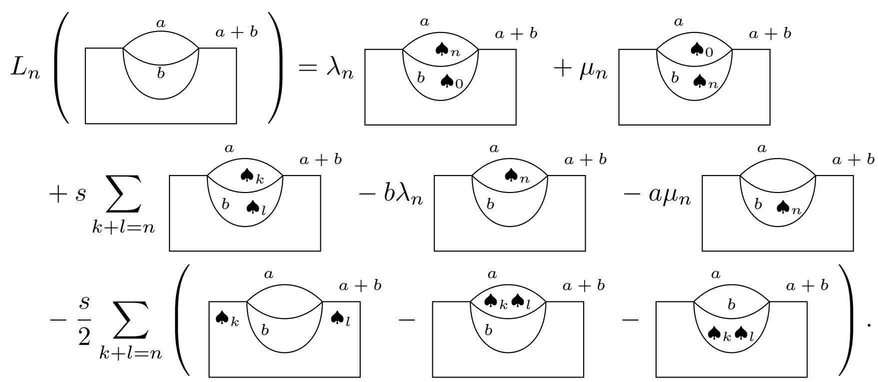
Lemma 4.13. *The foams:*



induce $\mathfrak{W}_{-1}$-equivariant morphisms:



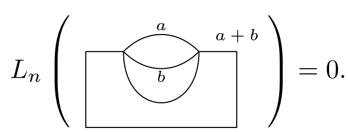
Proof.



Let $\spadesuit^a$ denote a decoration on the facet of thickness a. The computation between the parenthesis can be summarized by

$$\sum_{k+l=n} \spadesuit_k^{a+b}\spadesuit_l^{a+b} - \spadesuit_k^a\spadesuit_l^a - \spadesuit_k^b\spadesuit_l^b = 2\sum_{k+l=n} \spadesuit_k^a\spadesuit_l^b,$$

using the identity $\spadesuit_i^{a+b} = \spadesuit_i^a + \spadesuit_i^b$. Thus,

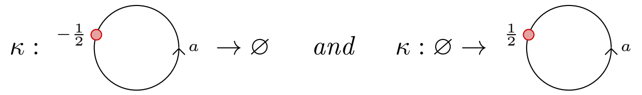


□

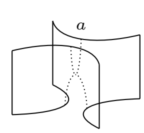
The proofs of the following lemmas are analogous.

Lemma 4.14. *The foams*

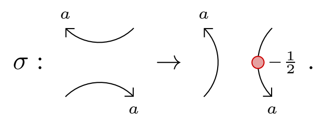
induce $\mathfrak{W}_{-1}$-equivariant morphisms



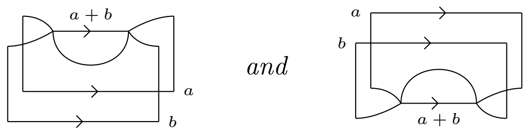
Lemma 4.15. *The foam*



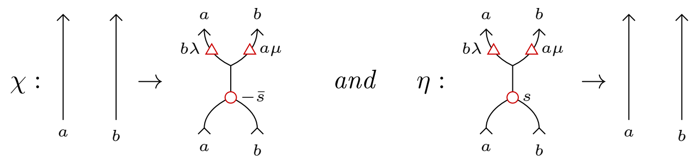
induces an $\mathfrak{W}_{-1}$-equivariant morphism



Lemma 4.16. *The foams*



induce $\mathfrak{W}_{-1}$-equivariant morphisms



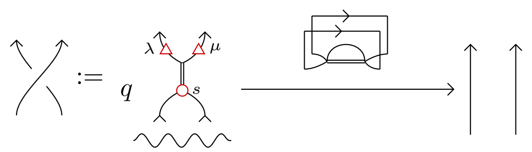
5. $\mathfrak{W}_{-1}$-equivariant Link Homology

Going forward, single edges (—), double edges (═), and bold edges (▬), will represent edges of thickness 1, 2, and 3 respectively in web diagrams.

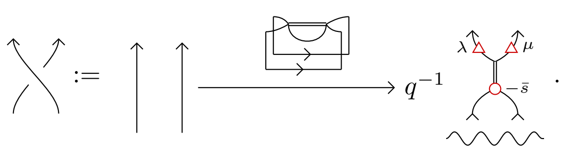
5.1. **Definition of a link homology.** Let D be a diagram of a link L in $\mathbb{R}^3$.

Definition 5.1. The complex $\mathrm{C}_N^{\mathfrak{W}}(L)$ is defined using the cube of resolutions construction associated to the braiding complexes

(37)



(38)



twisted with red dots to ensure that the differentials are $\mathfrak{W}_{-1}$-equivariant. The underlined terms are in cohomological degree 0. The cohomology $\overline{\mathrm{KRW}}_N(L, \mathrm{R}) := H^*(\mathrm{C}_N^{\mathfrak{W}}(L, R))$ is equipped with the action of $\mathfrak{W}_{-1}$.

Remark 5.2. It is easy to see that the action on the state spaces for webs appearing in $\mathrm{C}_2^{\mathfrak{W}}$ do not depend on the parameters. This is because $N = 2$ closed webs are isomorphic to webs without trivalent vertices via certain $N = 2$ foams, which foams can in turn be used to construct standard bases for state spaces [KW23, Section 3]. Direct computation shows that when the twists in the above braiding complexes are added, the action on the standard bases do not depend on the values of the parameters, λ, μ, and s. For general N, state spaces are more complicated and the dependence on parameters is unknown.

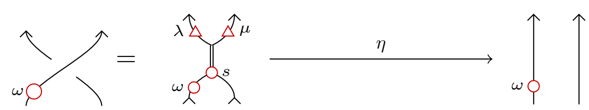
Remark 5.3. As in Example 4.7, there is a not a unique choice of red dots which make the differential equivariant. For example: the triangular dots can be placed on the two edges of thickness one, the hollow circles can be arranged on both the top and bottom edges of the same web, or the hollow dot can be replaced by a more complex dot which contributes cross-terms to both of the thickness one edges. In each of these cases, the twists no longer cancel out the parameter dependence and give non-isomorphic homology theories in the relative homotopy category. The twists defined above are the simplest option and the one which we analyze in the rest of the paper.

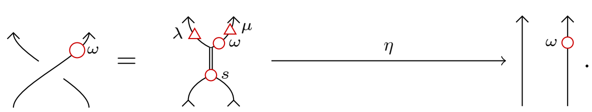
Theorem 5.4. *The chain complex* $\mathrm{C}_N^{\mathfrak{W}}(-)$ *is invariant under framed isotopy in the relative homotopy category.*

The rest of the section is dedicated to proving the theorem by showing invariance under Reidemeister moves.

5.2. **The sliding lemmas.** We begin with a technical lemma which is essential in the remainder of the section. The proof below is a clarification and adaptation of similar lemmas that appear in [KR16] and [QRSW23].

Lemma 5.5. *For any* $\omega \in \mathrm{R}$*, there is an isomorphism in the relative homotopy category between the complexes*

(39) ${}_xC :=$ 

(40) $C^y :=$ 

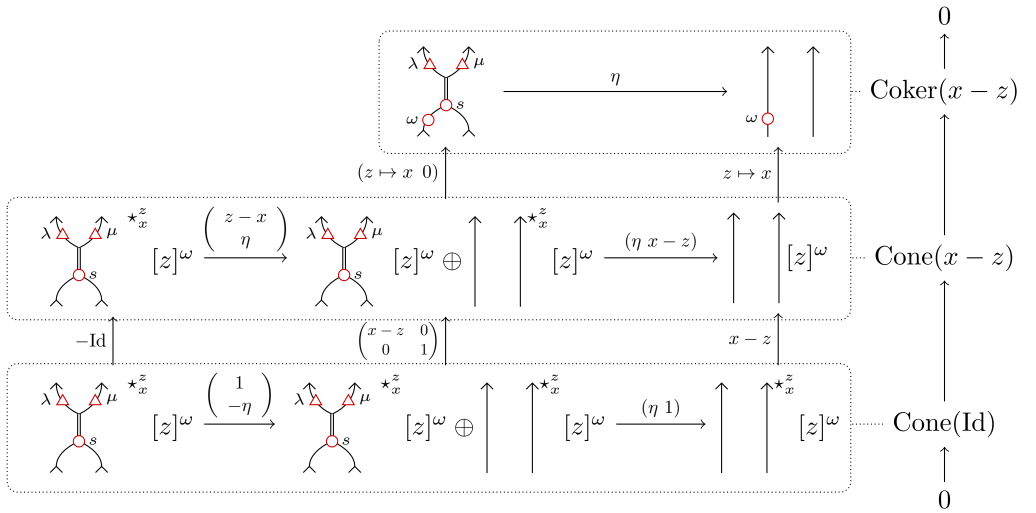
Proof. Let the left and right facets of η be called the x and y facets respectively. The map χ (see Lemma (4.16)) is a chain homotopy between the decoration maps x and y, but it cannot be twisted into equivariance. Our solution is to transfer twists from the x facet to the y facet via an intermediate formal variable z. To facilitate this, extend the base ring from R_N to $\mathrm{R}_N[z] := \mathrm{R}_{N,1}$, and the state space S of any web to $S[z] := S \otimes_{\mathrm{R}_N} \mathrm{R}_{N,1}$. The complexes ${}_xC[z]$ and $C^y[z]$ are defined as termwise extensions of ${}_xC$ and C^y.

Let $S[z]^\gamma$ denote the representation twisted by $L_n \mapsto -\gamma(n+1)z^n$, analogous to the red triangle, and let $S^{\star_x^z}[z]$ denote the one twisted by $L_n \mapsto -h_n(x,z)$, analogous to the solid red dot.

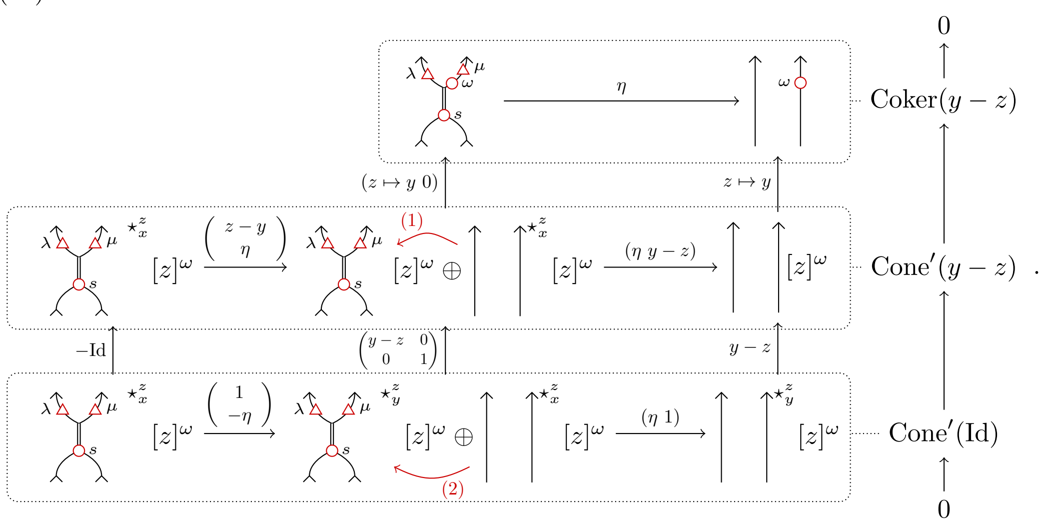
The variable z can be imagined as belonging to a cone point that every facet is connected to. This idea motivates the following algebraic construction. The map $C^{\star_x^z}[z]^\omega \xrightarrow{x-z} C[z]^\omega$ given by the difference of the x decoration map and multiplication by z is equivariant and its cokernel is equivariantly isomorphic to the original complex ${}_xC$. That the $\star$ twist makes the map $x-z$ equivariant while the ω twist on z corresponds to the ω twist in the cokernel. Similarly, $C^y \cong \mathrm{Coker}(C^{y\star_y^z}[z]^\omega \xrightarrow{y-z} C^y[z]^\omega)$. In the homotopy category, cokernels are isomorphic to cones and the extra room allows for additional twists on $\mathrm{Cone}(y-z)$ that make the isomorphism with $\mathrm{Cone}(x-z)$ corresponding to the chain homotopy χ equivariant.

This will prove that ${}_xC \cong C^y$.

(41)



(42)

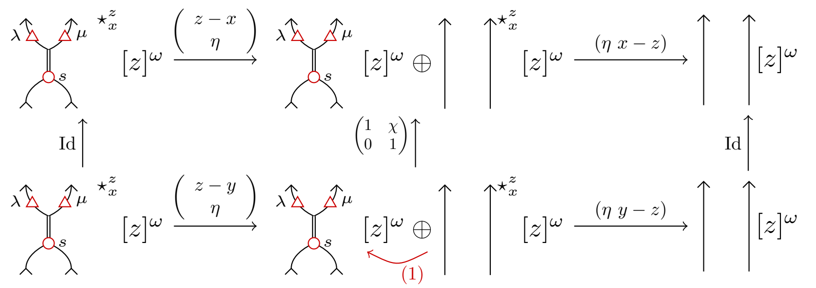


The red arrows indicate additional twists on the direct sum objects defined by

$$(1) : L_n \mapsto (h_n(x,z) - h_n(x,y)) \begin{pmatrix} 0 & 0 \\ 0 & \chi \end{pmatrix} \tag{43}$$

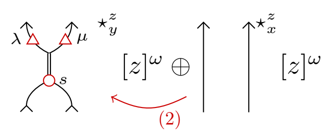
$$(2) : L_n \mapsto \frac{h_n(y,z) - h_n(x,z)}{x-y} \begin{pmatrix} 0 & 0 \\ 0 & \chi \end{pmatrix} \tag{44}$$

The additional twists on $\mathrm{Cone}'(y-z)$ make the following isomorphism to $\mathrm{Cone}(x-y)$ equivariant:

(45) 

∎

Remark 5.6. The twisted action of $\mathfrak{W}_{-1}$ on an element $F = F_1 + F_2$ in the state space of the direct sum

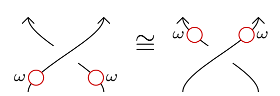
(46) $M =$ 

is given explicitly by

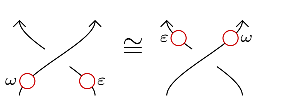
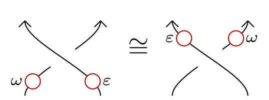
$$L_n \cdot_M F = L_n \cdot F - (\lambda(n+1)x^n + \mu(n+1)y^n)F_1 - (n+1)sx^nF_1 - h(z,y)F_1$$
$$- h_n(z,x)F_2 + \frac{h_n(y,z) - h_n(x,z)}{x-y}\chi \circ F_2.$$

Let $\omega, \varepsilon, \lambda$, and μ be elements of R. The following lemmas are consequences of dot migration.

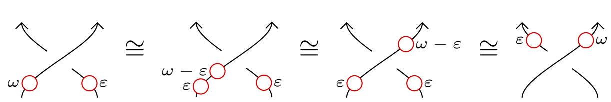
Lemma 5.7. *There is an isomorphism in the relative homotopy category:*

(47) 

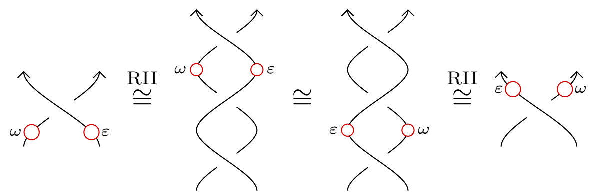
Corollary 5.8. *There are isomorphisms in the relative homotopy category:*

(48)  *and* 

Proof. The first isomorphism of (48) is deduced from Lemmas 5.5, 5.7, and the properties of red dots:

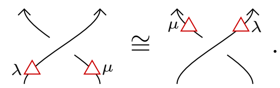
(49) 

To prove the second isomorphism in (48) apply Proposition 5.13, which does not rely on it.

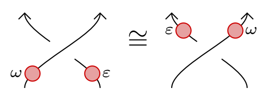
(50)


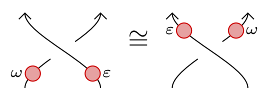
∎

Corollary 5.9. *There is an isomorphism in the relative homotopy category:*

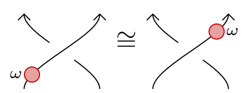


Lemma 5.10. *Let ω and ε be elements of R, there are isomorphisms in the relative homotopy category:*

(51)


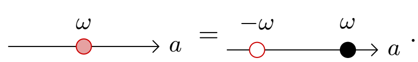
and


Proof. It suffices to prove the relation

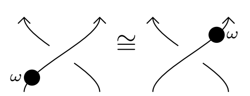
(52)


The rest of the proof follows as in Corollary 5.8.

To avoid dealing with the $N-1$ variables in this twist, introduce a new twist $L_n \mapsto -\sum_{k=0}^{n} p_k^{(e)} P_{n-k}$ recorded diagrammatically by a black dot. Since $P_{n-k} = p_{n-k}^{(e)} + \hat{p}_{n-k}^{(e)}$,



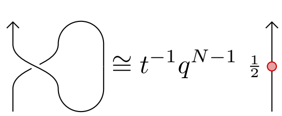
Lemma 5.5 implies that (52) follows from

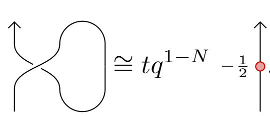
(53)


The proof of (53) is identical to the proof of Lemma 5.5 with $S[z]^\gamma$ redefined using the twist $L_n \mapsto -\gamma \left(\sum_{j=0}^{n} z^j P_{n-j}\right) z^k$. ∎

5.3. **Reidemeister I.**

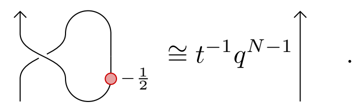
Proposition 5.11. *There are isomorphisms in the relative homotopy category:*

(54)


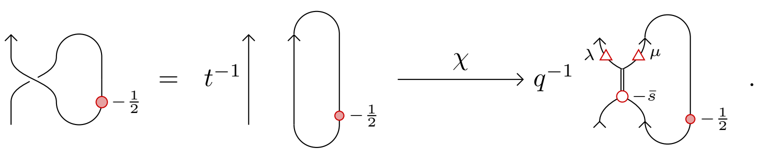
and


Proof. We only present the proof for the first isomorphism since the second uses the same ideas. Thanks to the Lemma 5.10, the existence of the following isomorphism is enough:

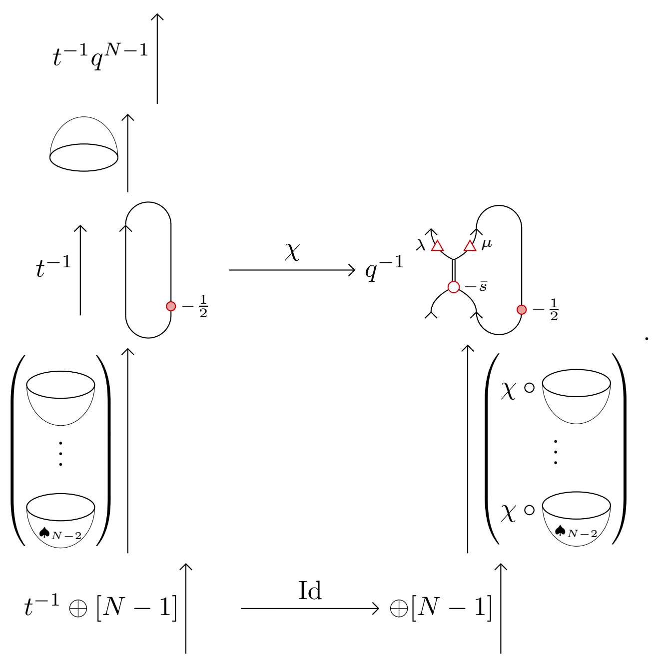
(55)



The first complex is defined to be:



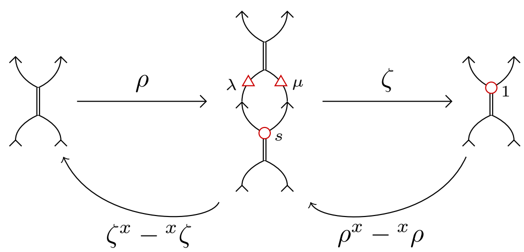
These two complexes embed in a short exact sequence of $\mathfrak{W}_{-1}$-equivariant complexes that splits after forgetting the $\mathfrak{W}_{-1}$-action:



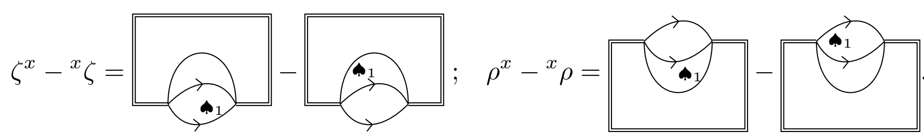
The splitting of the bottom complex does not respect the $\mathfrak{W}_{-1}$-structure, but by Corollary 2.9 there is an isomorphism between the top two rows in the relative homotopy category, since it is contractible. $\square$

5.4. **Reidemeister II.** The following result can be proved by straightforward computations using Section 4.3 and foam relations.

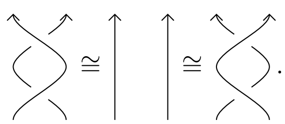
Lemma 5.12. *There is a short exact sequence of state spaces that splits when the $\mathfrak{W}_{-1}$-action is forgotten:*

(56)


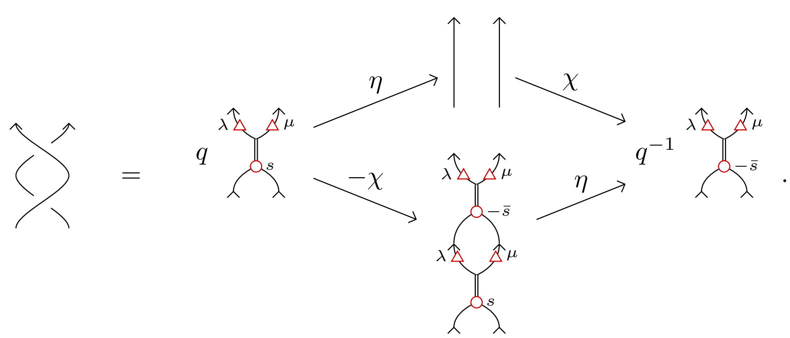
where the splitting maps are provided by:



Proposition 5.13. *There are isomorphisms in the relative homotopy category:*

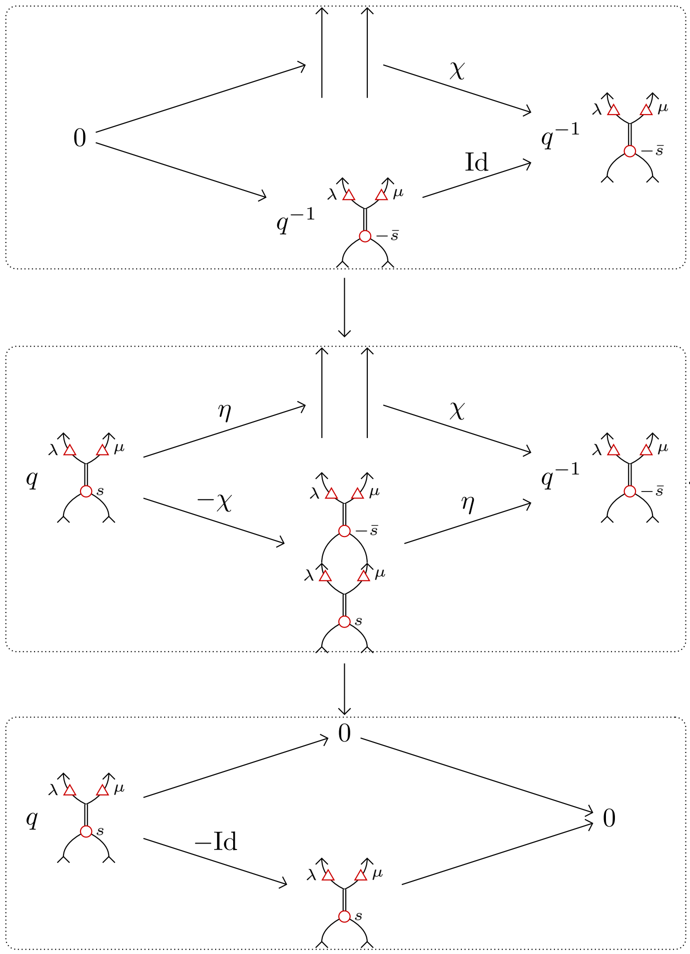
(57)


Proof. The constructions of both isomorphisms are the same so we only prove the first. By definition

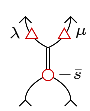
(58)


Thanks to Lemma 5.12 this complex fits into in a short exact sequence of complexes that splits when forgetting the $\mathfrak{W}_{-1}$-action.

(59)



The last complex is contractible, thus by using Corollary 2.9, the first and the second complexes are isomorphic in the relative homotopy category. Moreover, in the first complex, the terms

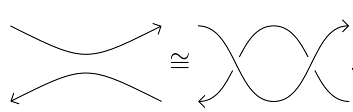
 ,

can be contracted resulting in the desired isomorphism. ☐

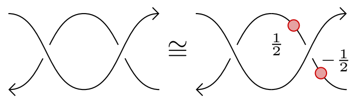
Note that the proof of the previous proposition does not require any result of Section 5.2.

Proposition 5.14. *There is an isomorphism in the relative homotopy category:*

(60)

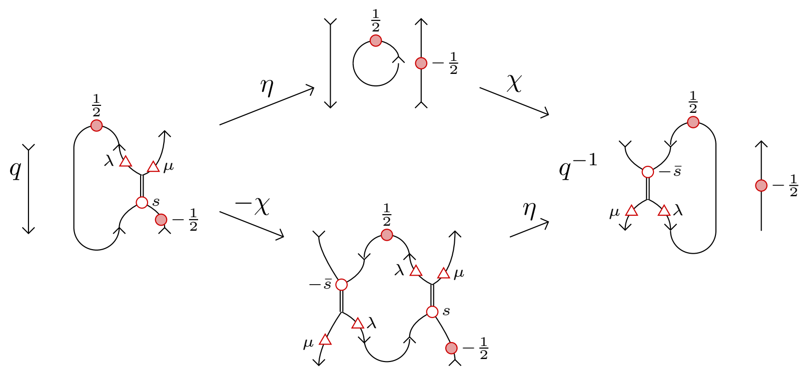


Proof. Additivity and sliding of dots in Lemma 5.10 implies the isomorphism



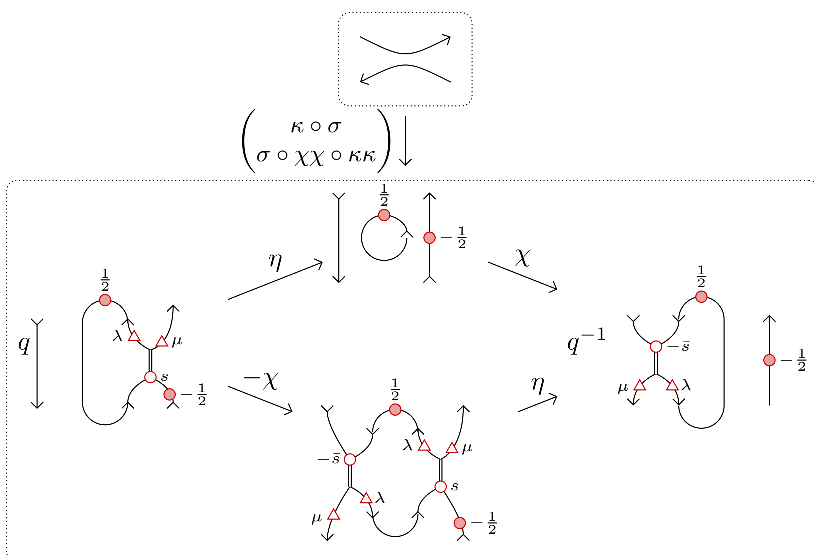
By definition, the second complex is given by:

(61)

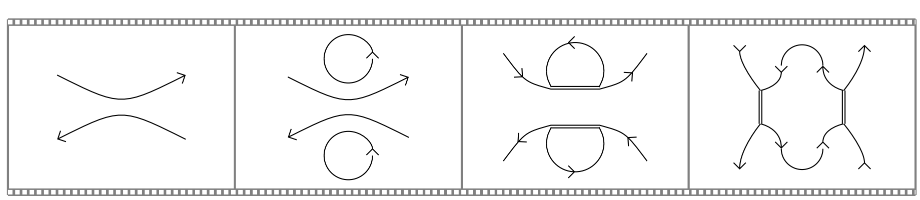


The following map is a homotopy equivalence when forgetting the $\mathfrak{W}_{-1}$-action, so it induces an isomorphism in the relative homotopy category:

(62)

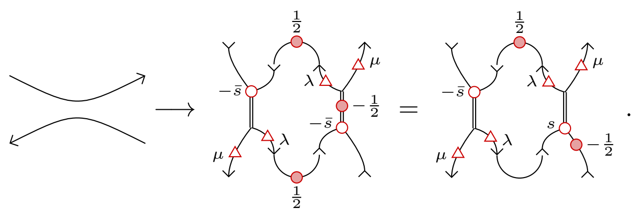


Let us check that the morphisms are indeed $\mathfrak{W}_{-1}$-equivariant. The map $\sigma\circ\chi\chi\circ\kappa\kappa$ is represented by the movie

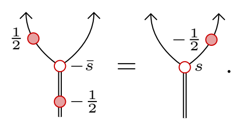


Lemmas (4.14), (4.15), and (4.16) show that the morphism $\sigma\circ\chi\chi\circ\kappa\kappa$ is equivariant with the following twist

$\sigma\circ\chi\chi\circ\kappa\kappa$:



The identity is due to Lemma 4.10. In this case, the relation takes the form

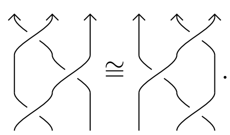


Thus, the map $\sigma\circ\chi\chi\circ\kappa\kappa$ in (62) is indeed $\mathfrak{W}_{-1}$-equivariant. The easier computation for $\kappa\circ\sigma$ is similar to the previous one and is thus omitted. ☐

5.5. **Reidemeister III.**

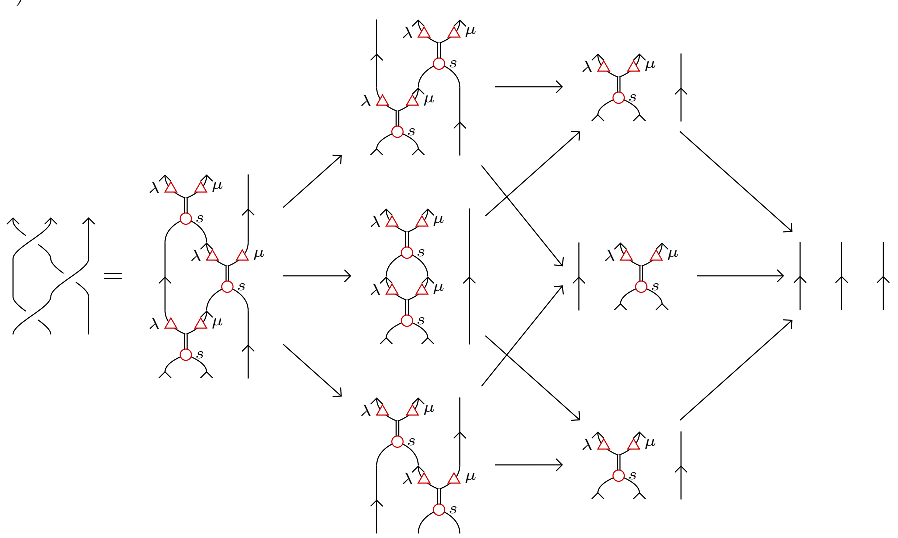
Proposition 5.15. *There are isomorphisms in the relative homotopy category:*

(63)



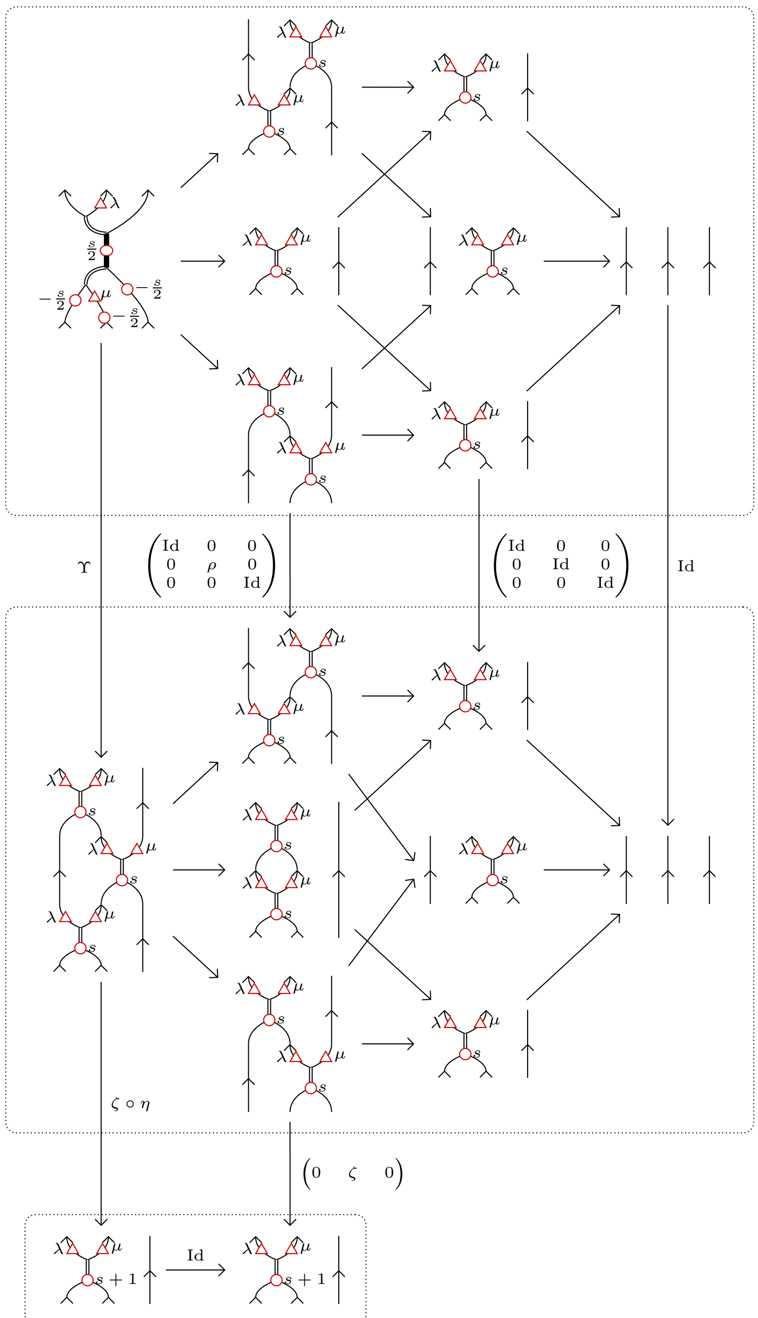
Proof. In the following proof, standard differentials and q-shifts are omitted for readability. The complex on the left-hand side in (63) is given by:

(64)

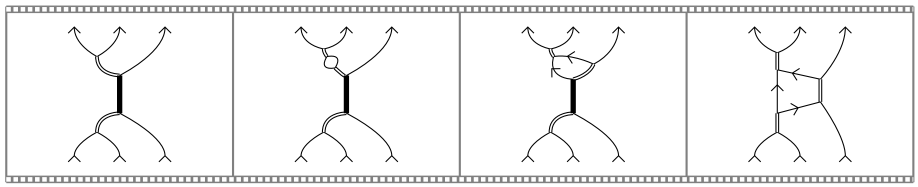


This complex fits in a short exact sequence of $\mathfrak{W}_{-1}$-equivariant complexes that splits after forgetting the $\mathfrak{W}_{-1}$ action.

(65)



The thickest edge of the web at the top left is of thickness 3. The map $\Upsilon := \theta \circ \alpha \circ \rho$ is the composition of a digon-cup, associativity, and the singular saddle. Its movie is:

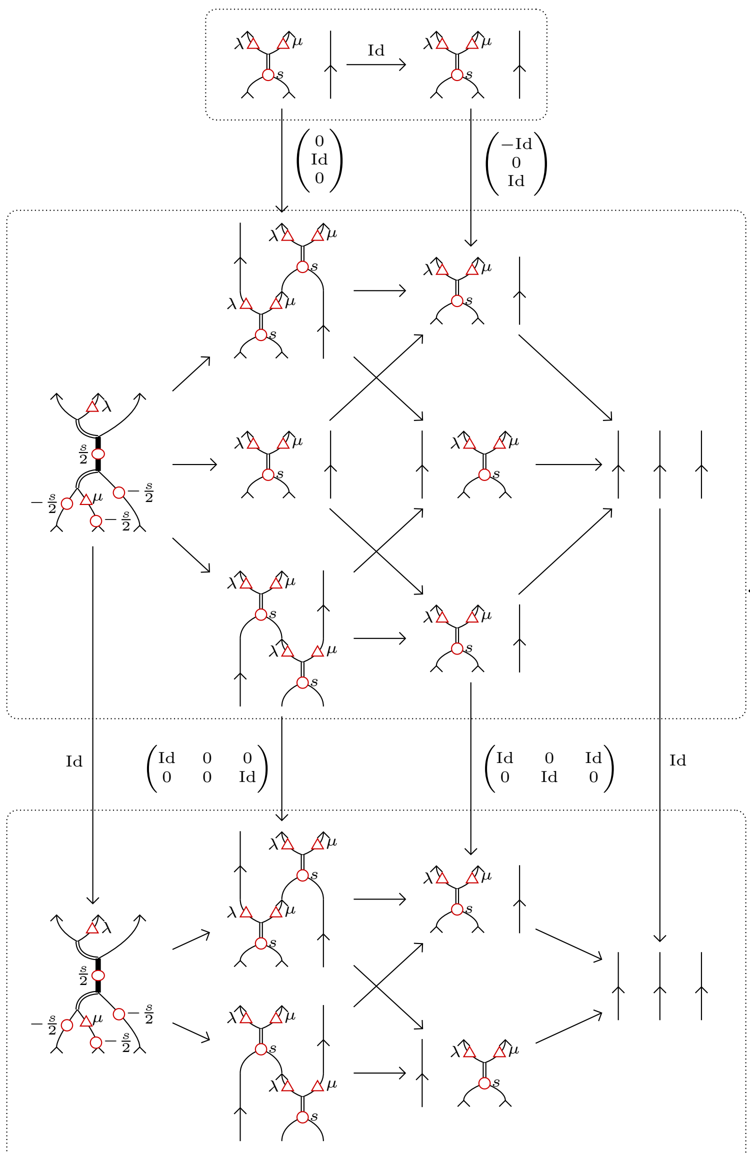


.

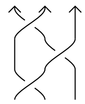
Since the bottom complex is acyclic, the two other complexes are isomorphic in the relative homotopy category.

There is a second short exact sequence of $\mathfrak{W}_{-1}$-equivariant complexes that splits when one forgets the $\mathfrak{W}_{-1}$ action.

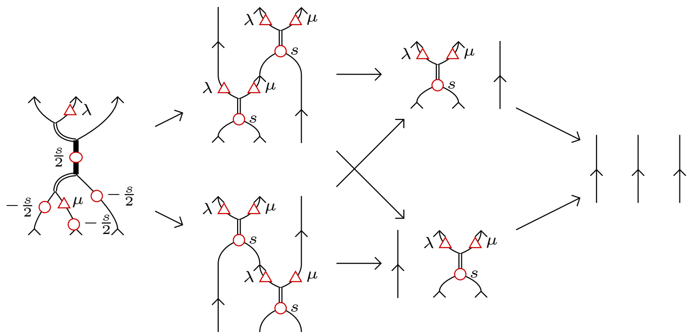
(66)



.

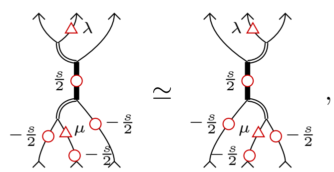
By the same argument as in (65),  is isomorphic to the following complex

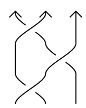
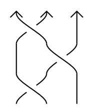
(67)



By the isomorphism

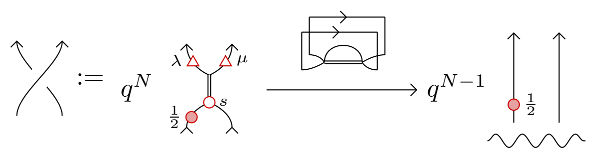
(68)



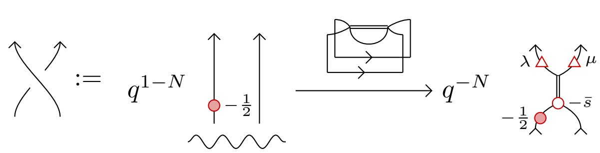
the complex (67) is left-right symmetric making  and  isomorphic to the same complex in the relative homotopy category. □

5.6. **Unframed Invariance.** The homology $\overline{\mathrm{KRW}}_N(\ .\ )$ is an invariant of framed link diagrams, however due to the grading shifts and twisting introduced by Reidemeister I moves, it is not naturally an unframed invariant. The braiding complexes can be modified slightly to correct this issue.

(69)



(70)



For a link L, let $\mathrm{KRW}_N(L, \mathrm{R})$ be the unframed ($\mathfrak{W}_{-1}$-equivariant) Khovanov–Rozansky $\mathfrak{gl}_N$-homology, that is the homology of the cube of resolution defined from the adjusted braiding complexes.

Corollary 5.16. *The unframed Khovanov–Rozansky $\mathfrak{gl}_N$-homology* $\mathrm{KRW}_N(L, \mathrm{R})$ *is an invariant of oriented links.*

6. Functoriality

The functoriality of $\mathfrak{gl}_N$ homology implies that link cobordisms induce R_N-linear maps between the homologies of the source and target links [ETW18]. However, there is no *a priori* reason for the induced maps to be $\mathfrak{W}_{-1}$-equivariant. In fact, the non-triviality of the action on cups, caps and saddles implies that induced maps will rarely be equivariant. Fortunately, the action on these building blocks is well-behaved.

Definition 6.1. Let rLinks be the category of links and cobordisms with scalar-labelled red dots on the components of links. Let $\mathrm{Mod}(\mathrm{R}_N)^{\mathfrak{W}_{-1}}_{g,p}$ denote the category of bigraded, projective, $(\mathfrak{W}_{-1}\#\mathrm{R}_N)$-modules with R_N-linear morphisms that are not necessarily $\mathfrak{W}_{-1}$-equivariant or homogeneous. The homology of a red dotted link is defined by adding solid red dot twists to the webs in the cube of resolutions.

Theorem 6.2. *There is a well-defined functor* $\mathrm{KRW}_N : \mathsf{rLinks} \to \mathrm{Mod}(\mathrm{R}_N)^{\mathfrak{W}_{-1}}_{g,p}$.

Proof. This follows immediately from [ETW18, Theorem 4.5] and the isotopy invariance of the $\mathfrak{W}_{-1}$-action on foams. □

Theorem 6.3. *Let K_1 and K_2 be knots decorated with red dots labelled λ_1 and λ_2 respectively and let*

$$C : K_1^{\lambda_1} \to K_2^{\lambda_2}$$

be a connected cobordism containing k Reidemeister I moves counted with sign. If $\lambda_2 - \lambda_1 = (\chi(C) - k)/2$, then

$$\overline{\mathrm{KRW}}_N(C) : \overline{\mathrm{KRW}}_N(K_1^{\lambda_1}) \to \overline{\mathrm{KRW}}_N(K_2^{\lambda_2})$$

is $\mathfrak{W}_{-1}$-equivariant. If $\lambda_2 - \lambda_1 = \chi(C)/2$, then

$$\mathrm{KRW}_N(C) : \mathrm{KRW}_N(K_1^{\lambda_1}) \to \mathrm{KRW}_N(K_2^{\lambda_2})$$

is $\mathfrak{W}_{-1}$-equivariant.

Proof. Every knot cobordism is a composition of cups, caps, saddles and Reidemeister moves. The maps associated to Reidemeister moves II and III are equivariant and Reidemeister I moves are equivariant up to a twist with positive and negative requiring the opposite correction. Examining the action on cups, caps, and saddles implies that the action depends only on the Euler characteristic and comparing with the correction for Reidemeister I moves shows that the required twists for both are precisely the solid red dot. □

Corollary 6.4. *Let L_1 and L_2 be links decorated with collections of red dots on each component labelled $\lambda_1 = (\lambda_1^i)$ and $\lambda_2 = (\lambda_2^j)$ respectively and let*

$$C : L_1^{\lambda_1} \to L_2^{\lambda_2}$$

be a connected cobordism containing k Reidemeister I moves counted with sign. If $\sum_i \lambda_2^i - \sum_j \lambda_1^j = (\chi(C) - k)/2$, then

$$\overline{\mathrm{KRW}}_N(C) : \overline{\mathrm{KRW}}_N(L_1^{\lambda_1}) \to \overline{\mathrm{KRW}}_N(L_2^{\lambda_2})$$

is $\mathfrak{W}_{-1}$-equivariant. If $\sum_i \lambda_2^i - \sum_j \lambda_1^j = \chi(C)/2$, then

$$\mathrm{KRW}_N(C) : \mathrm{KRW}_N(L_1^{\lambda_1}) \to \mathrm{KRW}_N(L_2^{\lambda_2})$$

is $\mathfrak{W}_{-1}$-equivariant.

An analogous statement for cobordisms with multiple components follows easily.

7. Polynomial Representations

In this section, we show that good coordinates can be given to the state spaces of simple foams by shifting the algebra of decorations by the inverse of a formal square root of a special polynomial. The action of L_m is extended using the power and chain rules:

$$L_m \cdot f^{-1/2} := \frac{1}{2} f^{-3/2}(L_m \cdot f).$$

In a previous paper by the second author, the action resricted to $\mathfrak{sl}_2$ was analyzed using this presentation [Roz26]. The algebra of decorations is decomposed as a $\mathfrak{W}_{-1}$-module, but a general analysis of the twisted modules is beyond the scope of the paper.

7.1. **Generalized theta webs.** Let $\mathbf{a} = (a_1, \dots, a_n)$ be an integer composition such that $\sum_i a_i = N$. For each k let $d_k := a_1 + \cdots + a_k$ be the kth partial sum. Define

$$t_{\mathbf{a}} := \prod_{k=1}^{n} \prod_{i=d_{k-1}+1}^{d_k} \prod_{j=d_k+1}^{N} (x_i - x_j).$$

Definition 7.1. Let $\theta_{\mathbf{a}}$ and $\Theta_{\mathbf{a}} \in \mathcal{F}_N(\theta_{\mathbf{a}})$ be the generalized theta web and foam associated to this composition, respectively:

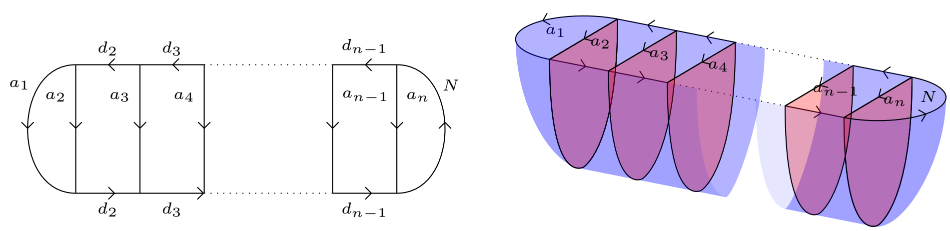


Figure 3. Theta web and foam

Proposition 7.2. *If $\lambda = \mu = 0$ and $s = 1/2$, then as $\mathfrak{W}_{-1}\#\mathrm{R}_N$-modules,*

$$\mathcal{F}_N(\theta_{\mathbf{a}}) \cong t_{\mathbf{a}}^{-1/2}\mathrm{R}_{\mathbf{a}}.$$

Proof. The state space of $\theta_{\mathbf{a}}$ is a rank one, free $\mathrm{R}_{\mathbf{a}}$-module generated by $\Theta_{\mathbf{a}}$, where $\mathrm{R}_{\mathbf{a}}$ acts as the algebra of decorations. The proposition is proved by comparing the $\mathfrak{W}_{-1}$-action on $\Theta_{\mathbf{a}}$ with the action on $t_{\mathbf{a}}^{-1/2}$.

The foam $\Theta_{\mathbf{a}}$ can be decomposed starting from the far right edge as a sequence of concentric cups, one for each facet of thickness a_k, and zips, one for each pair of thin facets with thicknesses a_k and d_{k-1}. After setting $\lambda = \mu = 0$, the action of L_m on the zip with thin facets a_k and d_{k-1} is multiplication by

$$-\overline{s} \sum_{i=0}^{m} p_i(x_1, \dots, x_{d_{k-1}}) p_{m-i}(x_{d_{k-1}+1}, \dots, x_{d_k}) \tag{71}$$

$$= -\overline{s} \sum_{i=d_{k-1}+1}^{d_k} \sum_{j=1}^{d_{k-1}} h_m(x_i, x_j). \tag{72}$$

The action on the cup of thickness a_k is multiplication by

$$\frac{1}{2}\sum_{i=0}^{m} p_i(x_{d_{k-1}+1},\dots,x_{d_k})\left(p_{m-i}(x_1,\dots,x_{d_{k-1}})+p_{m-i}(x_{d_k+1},\dots,x_N)\right) \tag{73}$$

$$= \frac{1}{2}\sum_{i=d_{k-1}+1}^{d_k}\left(\sum_{j=1}^{d_{k-1}} h_m(x_i,x_j)+\sum_{j=d_k+1}^{N} h_m(x_i,x_j)\right). \tag{74}$$

If $s = 1/2$, the first term of the cup contribution cancels with the zip contribution. The action of L_m on all zips and cups together is multiplication by

$$\frac{1}{2}\sum_{k=1}^{n}\sum_{i=d_{k-1}+1}^{d_k}\sum_{j=d_k+1}^{N} h_m(x_i,x_j). \tag{75}$$

On the other hand,

$$L_m\cdot(x_i-x_j) = -(x_i^{m+1}-x_j^{m+1}) = -h_m(x_i,x_j)(x_i-x_j). \tag{76}$$

Thus the action on $t_{\mathbf{a}}^{-1/2}$ is

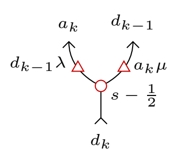
$$L_m\cdot t_{\mathbf{a}}^{-1/2} = L_m\cdot\left[\prod_{k=1}^{n}\prod_{i=d_{k-1}+1}^{d_k}\prod_{j=d_k+1}^{N}(x_i-x_j)^{-1/2}\right] \tag{77}$$

$$= \left[\frac{1}{2}\sum_{k=1}^{n}\sum_{i=d_{k-1}+1}^{d_k}\sum_{j=d_k+1}^{N} h_m(x_i,x_j)\right] t_{\mathbf{a}}^{-1/2}, \tag{78}$$

which agrees with the action on $\Theta_{\mathbf{a}}$. □

Remark 7.3. The leading term $t_{\mathbf{a}}^{-1/2}$ is the value of the foam evaluation formula on the non-closed foam $\Theta_{\mathbf{a}}$. Even though foam evaluation is well defined only for closed foams, the $\mathfrak{W}_{-1}$-action was defined so that it would commute with the local contributions from non-closed foams to the formula.

Remark 7.4. The condition on the parameters can be removed by adding red dot twists to $\theta_{\mathbf{a}}$ in appropriate places. At each of the below vertices of the theta web, the following twists cancel out the contribution of the λ, μ, and s parameters:



.

Additional solid red dots labeled λ at these vertices add twists to the polynomial ring of the form

$$\prod_{i=d_{k-1}+1}^{d_k}\prod_{j=d_k+1}^{N}(x_i-x_j)^{\lambda}.$$

Corollary 7.5. *Let $\varnothing$, U, and Θ denote the empty web, unknot, and standard theta web with thickness one thin facets. As $\mathrm{R}_N\#\mathfrak{W}_{-1}$-modules:*

- $\mathcal{F}_N(\varnothing)\cong\mathcal{F}_N(\theta_N)\cong\mathrm{R}_N$
- $\mathcal{F}_N(U)\cong\mathcal{F}_N(\theta_{1,N-1})\cong\prod_{i>1}(x_1-x_j)^{-1/2}\mathrm{R}_{1,N-1}$.
- $\mathcal{F}_N(\theta)\cong\mathcal{F}_N(\theta_{1,1,N-2})\cong\prod_{i>1}(x_1-x_i)^{-1/2}\prod_{j>2}(x_2-x_j)^{-1/2}\mathrm{R}_{1,1,N-2}$.

- $\mathcal{F}_N(\theta_{1^N}) \cong \prod_{i>j}(x_i - x_j)^{-1/2}\mathrm{R}_{1^N}$.

Proof. Examining equations (71) and (73) when $k = n$ shows that the facet with thickness N can be removed $\mathfrak{W}_{-1}$-equivariantly from $\Theta_{\mathbf{a}}$. Thus the webs $\varnothing, U$ and θ can be realized as special cases of the generalized theta web described above. □

7.2. **The algebra of decorations as a $\mathfrak{W}_{-1}$-module.** In the remainder of this section, we additionally require R to be a field of characteristic 0.

Recall that the action of $\mathfrak{W}_{-1}$ on R_{1^N} commutes with the symmetric group action permuting variables. Therefore, it is sufficient to decompose the full algebra of polynomials and then apply projections onto the symmetric subrings. Our strategy is to find generators for R_{1^N} as a $\mathfrak{W}_1$-module using Cartan-type harmonic polynomials then replace them with highest weight vectors to upgrade to a $\mathfrak{W}_{-1}$-decomposition.

Harmonic polynomials were first employed to find $\mathfrak{W}_1$-generators by [Nis96]. They were used to produce a full $\mathfrak{W}_{-1}$-decomposition for R_{1^2} in characteristic zero by [Tan01] and in positive characteristic by [CY21]. An $\mathfrak{sl}_2$-decomposition of R_{1^N} was previously found using standard theorems about nilpotent derivations [Bed24]. The contribution of this section is to combine these tools and provide more information on the basis of harmonic polynomials using the commuting S_N-action.

Definition 7.6 ([Hel00, Chapter III])**.** Let $\mathrm{Diff}(\mathrm{R}_{1^N})$ denote the algebra of polynomial differential operators under composition. The map $x_i \mapsto \frac{\partial}{\partial x_i}$ extends to an algebra homomorphism $\partial_{(-)} : \mathrm{R}_{1^N} \to \mathrm{Diff}(\mathrm{R}_{1^N})$ and the pairing $\langle p, q\rangle := (\partial_p q)(0)$ defines the *Fock inner product* on R_{1^N}.

Definition 7.7. For any $m \geq -1$, let

$$L_m^* := -\sum_{i=1}^{N} x_i \frac{\partial^{m+1}}{\partial x_i^{m+1}},$$

These operators satisfy the relation $[L_n^*, L_m^*] = (m-n)L_{n+m}^*$. Let $\mathfrak{W}_1^*$ denote the Lie algebra generated by L_m^* for $m \geq 1$.

Lemma 7.8 ([Nis96, Section 4])**.** *The operators L_m and L_m^* are adjoint with respect to the Fock inner product. The subspace $H_N := \bigcap_{m\geq 1} \ker L_m^*$ is finite-dimensional, orthogonal to $\mathfrak{W}_1(\mathrm{R}_{1^N})$, and a minimal generating subspace for R_{1^N} as a $\mathfrak{W}_1$-module.*

Remark 7.9. The elements of H_N are called *Cartan-harmonic polynomials*. With respect to the same inner product, ordinary harmonic polynomials are orthogonal to symmetric polynomials and a result analogous to Lemma 7.8 follows.

Proposition 7.10. *Let $\mathrm{R}[S_N/S_{k,N-k}]$ and $\mathrm{R}[S_k]$ denote the free* R*-modules generated by the sets $S_N/S_{k,N-k}$ and S_k respectively. The Cartan-harmonic polynomials contains a submodule isomorphic to*

$$\bigoplus_{k=0}^{N} \mathrm{R}[S_N/S_{k,N-k}] \otimes \mathrm{R}[S_k].$$

Proof. For any non-negative k, let $\mathrm{R}_{1^N}^k \subseteq \mathrm{R}_{1^N}$ denote the submodule generated by monomials each involving a collection of exactly k distinct variables. The R-module isomorphism $\mathrm{R}_{1^N} \cong \bigoplus_{k=0}^{N} \mathrm{R}_{1^N}^k$ is equivariant with respect to S_N and $\mathfrak{W}_1^*$. For any sequence of k distinct variables $x_{i_1}, \ldots, x_{i_k}$, the submodules

$$x_{i_1} \cdots x_{i_k} \mathrm{R}[x_{i_1}, \ldots, x_{i_k}] \subseteq \mathrm{R}_{1^N}^k$$

are fixed by $\mathfrak{W}_1^*$ and span $\mathrm{R}^k_{1^N}$. The space of leading coefficients is generated as an S_N-module by $e_k := x_1 \dots x_k$ and is isomorphic to $S_N/S_{k,N-k}$. This yields a $\mathfrak{W}_1^*$-equivariant isomorphism

$$\mathrm{R}^k_{1^N} \cong \mathrm{R}[S_N/S_{k,N-k}] \otimes e_k \mathrm{R}_{1^k},$$

with a trivial action on the left.

By Chevalley's theorem, there is an S_k-module isomorphism

$$e_k \mathrm{R}_{1^k} \cong \mathrm{R}[S_k] \otimes e_k \mathrm{R}_k.$$

The graded dimension of a free $\mathfrak{W}_1$-module on one generator agrees with the graded dimension of R_k in the first k degrees. Thus there are at least $k!$ $\mathfrak{W}_1$-generators and, by minimality, Cartan-harmonic polynomials. The action of $\mathfrak{W}_1$ commutes with S_N so the generators must form a copy of $\mathrm{R}[S_k]$. □

Lemma 7.11. *The only Cartan-harmonic polynomial in $e_k\mathrm{R}_k$ is e_k itself. Thus $e_k\mathrm{R}_k$ is indecomposible as a $\mathfrak{W}_0$-module.*

Proof. Let f be a homogeneous symmetric polynomial in $e_k\mathrm{R}_k$ such that the highest degree of x_1 is m. Suppose f contains a monomial

$$v := x_1^m x_2^{a_2} \cdots x_k^{a_k},$$

and thus $L^*_{m-1} f$ contains the monomial

$$w := x_1 x_2^{a_2} \cdots x_k^{a_k}.$$

If there is another monomial $u \neq v$ in f such that $L^*_{m-1}u$ contains the monomial w, u must be of the form $x_1 x_2^{a_2} \cdots x_i^{a_i+m-1} \cdots x_k^{a_k}$. The exponent $a_i + m - 1$ must be at least m, otherwise it would vanish under L^*_{m-1} and the maximality of m forces $a_i + m - 1$ to be at most m. Together these imply $a_i = 1$. Thus the exponents of v and u are the same up to a permutation so the symmetry of f guarantees that they have the same coefficient. Finally, this implies the coefficient of w in $L^*_{m-1}f$ is non-zero preventing f from being Cartan-harmonic. □

Corollary 7.12. *The base ring R_N is generated by the elementary symmetric polynomials as a $\mathfrak{W}_0$-module.*

Remark 7.13. Via the Thom isomorphism, the irreducible submodule $e_k\mathrm{R}_k$ is the ordinary cohomology of $MU(k)$ and its irreducibility corresponds to classical results about cohomology operations and characteristic classes.

Conjecture 7.14. *The submodule in Proposition 7.10 is the entire space of Cartan-harmonic polymomials.*

Remark 7.15. To prove the conjecture, it is sufficient to show that R_k-linear combinations of the Cartan-harmonic copies of $\mathrm{R}[S_k]$ are never Cartan-harmonic. When $N = 2$ all that has to be shown is that there are no Cartan-harmonic anti-symmetric polynomials. In general this is more complicated because higher dimensional representations of S_k appear with multiplicity in different degrees. In [Nis96], Cartan-harmonic polynomials were computed explicitly for $N = 3, 4$ with dimensions agreeing with the conjecture.

Lemma 7.16. *The polynomial $(x - y)xy$ is the only Cartan-harmonic polynomial in the submodule $(x - y)xy\mathrm{R}_2$.*

Proof. Let f be a degree n homogeneous element of $(x-y)xy\mathrm{R}_2$. Any such anti-symmetric polynomial can be written as a sum

$$f = \sum_{i=1}^{\lceil n/2 \rceil} a_i(x^{n-i}y^i - x^i y^{n-i}).$$

then $L_1^* f$ can be written as the sum

$$L_1^* f = \sum_{i=1}^{\lceil n/2 \rceil} a_i(n-i)(n-i-1)(x^{n-i-1}y^i - x^i y^{n-i-1}) \tag{79}$$

$$+ \sum_{i=1}^{\lceil n/2 \rceil} a_i i(i-1)(x^{n-i}y^{i-1} - x^{i-1}y^{n-i}). \tag{80}$$

Let k be the largest k such that $a_k \neq 0$.

If n is even, then $n-k-1 \neq k$ so $(x^{n-k-1}y^k - x^k y^{n-k-1}) \neq 0$. The coefficient of $(x^{n-k-1}y^k - x^k y^{n-k-1})$ in $L_1^* f$ is precisely $a_k(n-k)(n-k-1)$. Since f is divisible by $(x-y)xy$, k cannot be equal to n and the coefficient is non-zero. Thus $L_1^* f \neq 0$, so f is not Cartan harmonic.

If n is odd and $k \neq (n-1)/2$, then the same argument applies since $n-k-1 \neq k$. Now assume that $k = (n-1)/2$. In this case $(x^{n-k-1}y^k - x^k y^{n-k-1}) = 0$ so consider the next term in the expansion. The coefficient of $(x^{k+1}y^{k-1} - x^{k-1}x^{k+1})$ in $L_1^* f$ is

$$a_{k-1}(k+2)(k+1) + a_k k(k-1).$$

The coefficient of $(x^k y^{k-1} - x^{k-1}y^k)$ in $L_2^* f$ is

$$a_{k-1}(k+2)(k+1)k - a_k(k+1)k(k-1).$$

Unless $a_{k-1} = 0$ or $k = 1$, coefficients cannot both be zero so at least one of $L_1^* f$ and $L_2^* f$ is non-zero. As long as $f \neq (x-y)xy$, $k > 1$. Thus if $a_{k-1} = 0$ then $L_1^* f$ is guaranteed to be non-zero. We conclude that no $f \in (x-y)xy\mathrm{R}_2$ is Cartan-harmonic except for $(x-y)xy$ itself. $\square$

Proposition 7.17. *As a $\mathfrak{W}_{-1}$-module,* R_{1^N} *is a direct sum of* $\dim(H_N) - 1$ *indecomposible $\mathfrak{W}_{-1}$-submodules generated by highest weight vectors.*

Proof. The operator L_{-1} is adjoint to multiplication by p_1 so in analogy with Lemma (7.8) there is a ring isomorphism

$$\mathrm{R}_{1^N} \cong (\ker L_{-1})[p_1].$$

Cartan-harmonic generators which are not killed by L_{-1} are in $\mathfrak{W}_{-1}$-extensions with other $\mathfrak{W}_1$-submodules. To produce split $\mathfrak{W}_{-1}$-extensions, the Cartan-harmonic $\mathfrak{W}_1$-generators should be replaced with the constant terms in their p_1 polynomial expansions via the quotient map

$$R_{1^N} \twoheadrightarrow R_{1^N}/p_1 R_{1^N} \cong \ker L_{-1}.$$

The polynomial p_1 is Cartan-harmonic and thus cannot be replaced by a highest weight generator. No other Cartan-harmonic polynomial is divisible by p_1 and therefore a highest weight replacement is possible for all other generators. Suppose $q \cdot p_1$ were Cartan-harmonic for some homogeneous polynomial q. The commutation relation in $\mathfrak{W}_{-1}^*$ implies

$$L_1^*(p_1 q) = 2L_0^* q + p_1 L_1^* q.$$

Since the operator $2L_0^* = 2L_0$ multiplies polynomials by their degree, $L_1^*(qp_1)$ vanishes when

$$p_1 L_1^* q = -(\deg q)q.$$

This only happens if q is 1 or divisible by p_1. Repeating the commutation relation for $p_1 L_1^* q$ shows that if $q \neq 1$, it would be infinitely divisible by p_1 which is impossible. Thus $L_1^*(qp_1) \neq 0$ for any $q \neq 1$ and there are no Cartan-harmonic polynomial multiples of p_1. In particular, this implies that all other Cartan-harmonic generators have non-zero images under the quotient map onto $\ker L_{-1}$.

The $\mathfrak{W}_1$-submodule generated by p_1 is in extension with the trivial submodule generated by 1. Together these form a $\mathfrak{W}_{-1}$-direct summand. The remaining $\mathfrak{W}_1$-generators can be replaced with highest weight generators forming $\mathfrak{W}_{-1}$-direct summands. Since each highest weight replacement is accomplished by subtracting terms belonging to other $\mathfrak{W}_1$-modules, the corresponding quotient modules are isomorphic as $\mathfrak{W}_1$-modules. In particular they are indecomposible. □

Corollary 7.18. *In the case $N = 2$, $\ker L_{-1} \cong \mathrm{R}[(x-y)]$ and the highest weight $\mathfrak{W}_{-1}$-generators are $1, (x-y), (x-y)^2$, and $(x-y)^3$.*

7.3. **Twisted representations.** Analyzing the twisted representations of the form $t_{\mathbf{a}}^{-1/2}\mathrm{R}_{\mathbf{a}}$ is harder since the inner product and adjoint Witt operators do not have obvious extensions even in the simplest case $N = 2$. However, the highest weight structure of R_{1^2} is simple. It was shown in [Roz26] that the highest weight submodule of $(x-y)^\lambda \mathrm{R}_{1^2}$ is $(x-y)^\lambda \mathrm{R}[(x-y)]$ where $\lambda \in \frac{1}{2}\mathbb{Z}$.

Direct computation shows that

$$L_1^2(x-y)^\lambda = \lambda L_1\left((x+y)(x-y)^\lambda\right) \tag{81}$$

$$= \lambda\left((x^2+y^2) + \lambda(x+y)^2\right)(x-y)^\lambda \tag{82}$$

$$= \lambda\left((1+\lambda)(x^2+y^2) + 2\lambda xy)\right)(x-y)^\lambda. \tag{83}$$

On the other hand,

$$L_2(x-y)^\lambda = \lambda(x^2+xy+y^2)(x-y)^\lambda. \tag{84}$$

These are linear combinations precisely when $2\lambda^2 = \lambda(1+\lambda)$. That is when $\lambda = 0, 1$.

This implies that unless $\lambda = 0, 1$, the $\mathfrak{sl}_2$-submodule generated by $(x-y)^\lambda$ is in $\mathfrak{W}_{-1}$-extension with the one generated by $(x-y)^{\lambda+2}$. In particular, this shows that the homology of the unknot, $(x-y)^{-1/2}\mathrm{R}_{1^2}$ is a direct sum of the two $\mathfrak{W}_{-1}$-submodules generated by $(x-y)^{-1/2}$ and $(x-y)^{1/2}$.

7.4. **Lee and Bar-Natan Deformations.**

7.4.1. *Universal Lee theory.* The (X^2+t)-deformation otherwise known as universal Lee theory is recovered from R_2-linear homology by setting $E_1 = x + y$ to zero. The above results suggest that the Lee deformation of the unknot can be identified as the space of highest weight vectors $\mathrm{R}[(x-y)]$ and the deformation of the base ring is the subspace of symmetric, that is even, elements: $\mathrm{R}[(x-y)^2]$. In fact, the functoriality of the Witt representation shows that this identification is natural with respect to link cobordisms.

Proposition 7.19. *The kernel of L_{-1} on KRW_2 is naturally isomorphic to the universal Lee deformation of Khovanov homology.*

Proof. Suppose Γ is a closed $\mathfrak{gl}_2$-web. Let G be the collection of unknots obtained by removing thickness two edges from Γ and forgetting orientations. As a graded R_2-module, $\mathcal{F}_2(\Gamma)$ is isomorphic to $\mathcal{F}_2(G)$ [KW23]. In particular, it is a finite tensor product of grading shifts of $\mathrm{R}_{1,1}$ over the base ring R_2. The copies of $\mathrm{R}_{1,1}$ are the algebras of decorations of closed thickness one subgraphs of Γ where the assignment of variables x and y to edges alternates each time the orientation changes.

By Remark 5.2, we are free to specialize $\lambda = \mu = 0$ and $s = 1/2$ without affecting the homology. With these restrictions, the $\mathfrak{W}_{-1}$ action on a foam is identical to the action on the foam with thickness 2 facets removed. Crucially, the action of L_k on any basic foam is multiplication by $h_k(x, y)$ where x and y are respectively the variable on and off the same thickness one edge of G. The overall twist is a sum of these operators and can be represented by a leading factor of the form $(x - y)^l$ on the corresponding copy of $\mathrm{R}_{1,1}$ that appears in the state space. The twists in the braiding complex behave in the same way, adding additional twists of the form $(x - y)^{\pm 1/2}$.

Thus, the only $\mathfrak{W}_{-1}$ representations appearing in cube of resolution are tensor products of $(x - y)^l \mathrm{R}_{1,1}$ for various values of l. Moreover, these values agree with the usual grading shifts present in the Khovanov complex. The analysis of the previous section identifies the highest weight vectors of these modules with the Lee state spaces of the corresponding webs. Since highest weight vectors are preserved by equivariant maps, the subspaces are preserved by maps induced by cobordisms, in particular the differentials. □

The highest weight subspace is not a $\mathfrak{W}_{-1}$-submodule and $\mathrm{R}_{1,1}/E_1\mathrm{R}_{1,1}$ is not a $\mathfrak{W}_{-1}$-quotient module. However, we can still ask how the individual differential operators L_k act on the quotient vector space with the highest weight vectors chosen as consistent representatives.

Adding triangular twists with non-zero λ and μ makes this action more interesting. Let ε_x be a formal variable of degree 2 with an action $L_k \cdot \varepsilon_x := (k+1)x^k \varepsilon_x$ so that the module $\varepsilon_x \mathrm{R}_{1,1}$ is the state space of a thickness one unknot with a triangular dot labelled -1.

Remark 7.20. Recall that the hollow dot on a thickness one facet agrees with the triangle. Therefore, the twisted state space should behave formally like $(x - x)\mathrm{R}[x, y]$ with the action

$$L_k \cdot (x - x) = -(x^{k+1} - x^{k+1}) = -h_k(x, x)(x - x) = -(k + 1)x^k(x - x).$$

Thus, the triangular twist ε_x can be thought of as a formally non-zero deformation of $(x - x)$.

Suppose a theta web is decorated by triangular dots on the thickness one edges with labels λ and $-\lambda$ so that the grading shift of the state space is unchanged. Let $\varepsilon_x^\lambda \varepsilon_y^{-\lambda}$ be abbreviated by $\varepsilon_x/\varepsilon_y)^\lambda$ so that the twisted state space is $(\varepsilon_x/\varepsilon_y)^\lambda \mathrm{R}_{1,1}$. The action on the generator is

$$L_k \cdot (\varepsilon_x/\varepsilon_y)^\lambda = \lambda(k + 1)(x^k - y^k)(\varepsilon_x/\varepsilon_y)^\lambda \tag{85}$$

$$= \lambda(k + 1)h_{k-1}(x, y)(x - y)(\varepsilon_x/\varepsilon_y)^\lambda, \tag{86}$$

where $h_{-1} := 0$. The action on highest weight vectors is

$$L_k \cdot (x-y)^n(\varepsilon_x/\varepsilon_y)^\lambda = \tag{87}$$

$$\Big[\lambda(k+1)h_{k-1}(x,y)(x-y) - nh_k(x,y)\Big](x-y)^n(\varepsilon_x/\varepsilon_y)^\lambda. \tag{88}$$

When E_1 is set to zero, the polynomial $h_k(x,y)$ vanishes for odd k and is represented by $2^{-k}(x-y)^k$ for even k. Therefore, when k is odd,

$$L_k \cdot (x-y)^n(\varepsilon_x/\varepsilon_y)^\lambda = 2^{-k+1}\lambda(k+1)(x-y)^{n+k}(\varepsilon_x/\varepsilon_y)^\lambda, \tag{89}$$

and when k is even,

$$L_k \cdot (x-y)^n(\varepsilon_x/\varepsilon_y)^\lambda = -2^{-k}n(x-y)^{n+k}(\varepsilon_x/\varepsilon_y)^\lambda. \tag{90}$$

In this presentation, an additional quotient by $(x-y)^2$ to obtain ordinary Khovanov homology, simply makes L_k act by 0 for $k \geq 2$. The action of L_0 multiplies by degree and L_1 multiplies by $2\lambda(x-y)$.

Link homology can be made sensitive to this strange alternating behavior by changing the positions of the dots in the braiding complexes. Instead of the usual configuration, place the two triangular dots on the two unlinked edges and set $\mu = -\lambda$. The result is that twists in the braiding complex and thus Reidemeister I moves introduce a twist by $(\varepsilon_x/\varepsilon_y)^{\pm\lambda}$ in addition to the original twist $(x-y)^{\pm 1/2}$.

7.4.2. *Bar-Natan homology.* The (X^2-hX)-deformation, otherwise known as Bar-Natan homology, also appears in this picture by setting $E_2 = xy$ to zero. The submodule $xy\mathrm{R}_{1,1}$ is not invariant under L_{-1}, but it is a $\mathfrak{W}_0$-submodule, immediately proving the following proposition.

Proposition 7.21. *The Bar-Natan deformation of Khovanov homology is a $\mathfrak{W}_0$-module.*

The state space of the unknot, $\mathrm{R}_{1,1}/E_2\mathrm{R}1,1$, consists of the irreducible submodules generated by $(x+y)$ and $(x-y)$. Representatives for the rest of the first submodule can be given by power sums x^k+y^k and the second by power differences x^k-y^k. The base ring appears as the submodule of power sums. The twisted modules $(x-y)^l\mathrm{R}_{1,1}$ can also be represented by shifting the power sums and differences by the leading factor $(x-y)^l$. For any twisting there are two $\mathfrak{W}_0$-submodules which are necessarily distinct since one is symmetric and one is antisymmetric. Moreover, this choice of representatives carries an $\mathfrak{sl}_2$ action. The results of [Roz26] show that as $\mathfrak{sl}_2$-modules, the pair of submodules are dual Vermas with highest weights corresponding to the twisting. It is easily seen that the $\mathfrak{sl}_2$ action is not natural with respect to cobordisms.

In fact, the $\mathfrak{W}_0$-splitting described above is also a splitting as $\mathrm{R}[E_1]$ modules. However, triangular twists change this. The twisted action can be represented by

$$L_k \cdot (x-y)^n(\varepsilon_x/\varepsilon_y)^\lambda = \tag{91}$$

$$\Big[\lambda(k+1)(x^k-y^k) - n(x^k+y^k)\Big](x-y)^n(\varepsilon_x/\varepsilon_y)^\lambda. \tag{92}$$

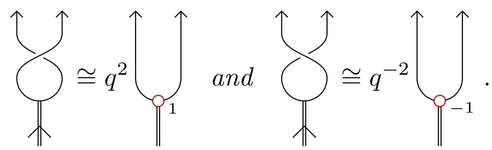
Thus the homology is rank two as a $\mathfrak{W}_0$-module and as a $\mathrm{R}[E_1]$-module in distinct ways. This breaks the symmetry in an interesting way and may be used to reveal further structure.

8. $(2, m)$ Torus Links

8.1. **Forktwist lemma.**

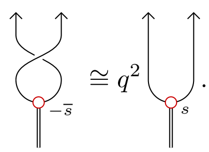
Lemma 8.1. *There are the following isomorphisms in the relative homotopy category:*

(93)

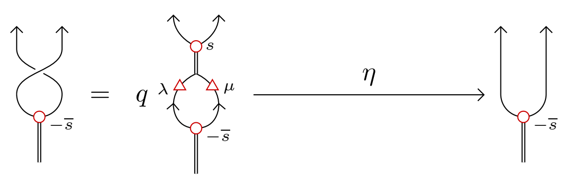


Proof. The construction of both isomorphisms uses the same ideas so we only present the proof for the first isomorphism. Thanks to Lemma 5.7 the existence of the following isomorphism is enough:

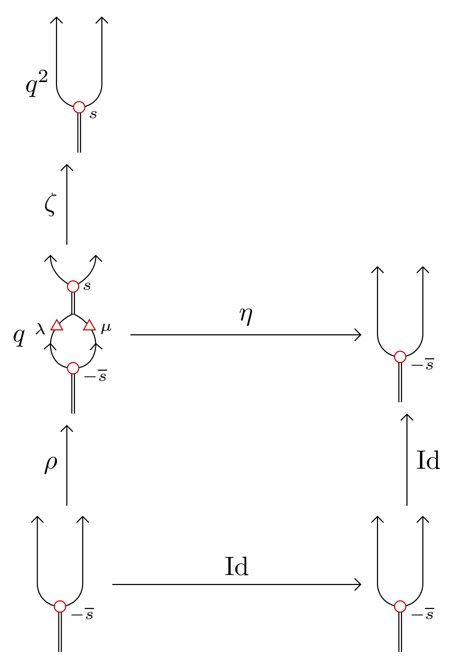
(94)



The first complex is defined to be:



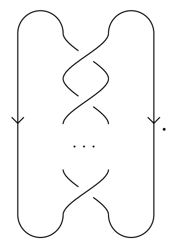
These two complexes fit in a short exact sequence of $\mathfrak{W}_{-1}$-equivariant complexes that splits after forgetting the $\mathfrak{W}_{-1}$ action:



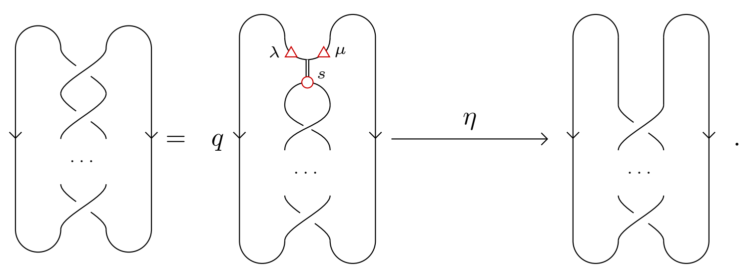
The vertical maps are given by the digon cup and digon cap. Since the bottom complex is contractible, Corollary 2.9 gives the desired isomorphism.

□

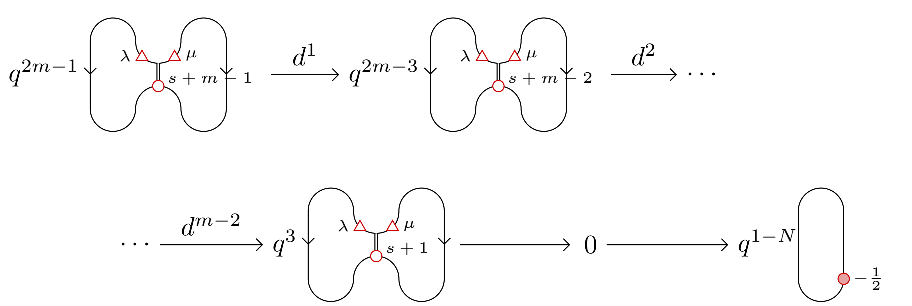
8.2. $(2,k)$ **torus Links.** Using a standard argument we compute the action on $T_{2,m}$:



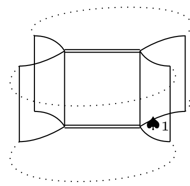
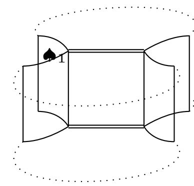
The following complex remains after resolving the top crossing only:

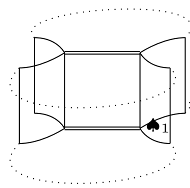
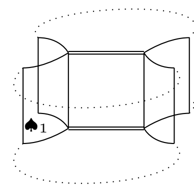


Applying fork twists 8.1 to the left term and induction on the right term, the complex below is obtained.



The differentials alternate between the following two maps:

(95) $d^{m-2i} =$  $-$ 

(96) $d^{m-2i-1} =$  $-$  $= 0$

The results of section 7 make the homology computation easy:

Theorem 8.2. *As a graded $\mathfrak{W}_{-1}\#\mathrm{R}_N$-module*

$$\overline{\mathrm{KRW}}^m_N(T_{2,m}) \cong \prod_{1<i\leq N} (x_1 - x_i)^{-1}\mathrm{R}_{1,N-1},$$

$$\overline{\mathrm{KRW}}^{m-j}_N(T_{2,m}) \cong 0 \textit{ for all odd } j,$$

$$\overline{\mathrm{KRW}}^{m-j}_N(T_{2,m}) \cong (x_1 - x_2)^{j-2}\mathrm{R}_{1,1,N-2}/(x-y)^{j-1}\mathrm{R}_{1,1,N-1} \textit{ for all even } 0 < j < m,$$

$$\overline{\mathrm{KRW}}^0_N(T_{2,m}) \cong \prod_{1<i\leq N} (x_1 - x_i)^{m-2}\mathrm{R}_{1,N-1} \textit{ if } m \textit{ is even.}$$

8.3. **The case $N = 2$.** The first goal is to describe all $\mathfrak{W}_{-1}\#\mathrm{R}_2$-equivariant maps $\mathrm{KRW}(\varnothing) \to \mathrm{KRW}(T^\lambda_{2,m})$. Unfortunately, the techniques of Section 7 do not carry over well to the twisted case because the inner product is not well defined when twists are negative or fractional. Even when twists are positive, the adjoints are not always well defined. Thus *ad hoc* methods and direct computation are necessary.

The twisted homology can be described easily when $N = 2$. We present the unframed version to avoid issues with counting Reidemeister moves.

Corollary 8.3. *As graded $\mathfrak{W}_{-1}\#\mathrm{R}_N$-modules*

$$\mathrm{KRW}^0_2(T^\lambda_{2,m}) \cong (x-y)^{\frac{\lambda+m}{2}-1}\mathrm{R}[x,y],$$

$$\mathrm{KRW}^{-j}_2(T^\lambda_{2,m}) \cong 0 \textit{ for all odd } j,$$

$$\mathrm{KRW}^{-j}_2(T^\lambda_{2,m}) \cong (x-y)^{\frac{\lambda+m}{2}+j-2}\mathrm{R}[x,y]/(x-y)^{\frac{\lambda+m}{2}+j-1}\mathrm{R}[x,y] \quad \textit{for all even } 0 < j < m,$$

$$\mathrm{KRW}^{-m}_2(T^\lambda_{2,m}) \cong (x-y)^{\frac{\lambda+m}{2}+m-2}\mathrm{R}[x,y] \textit{ if } m \textit{ is even.}$$

Any such R_N-linear map is determined by the image of 1 and $\mathfrak{W}_{-1}$-equivariance is guaranteed if the image is trivial module. When $N = 2$ the decomposition above agrees with the $\mathfrak{sl}_2$ decomposition in [Roz26] of the framed homology. It was shown that the only highest weight vectors are powers of $(x - y)$ and in particular the only trivial submodule is generated by $(x - y)^0$. Therefore the only maps which can be induced by cobordisms C are scalar multiples of the inclusion of symmetric polynomials into $(x-y)^{(\lambda+m)/2-1}\mathrm{R}[x,y]$. These only exist when $\lambda+m$ is even and $\chi(C)/2 = \lambda \leq 2-m$. When m is odd, so that $T_{2,m}$ is a knot, $\chi(C) = 1-2g$ and $m-3/2 \leq m-1 \leq g$ recovering the genus bound from the s-invariant.

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